Using Quandles and the Alexander-Conway Polynomial to Detect Causality

Joshua Piesner

October 19, 2021

Abstract

I examine whether the coloring invariant of specific Alexander quandles can be paired with the Alexander-Conway polynomial so that they can plausibly detect causality between two events in (2+1)-dimensional globally hyperbolic spacetime.

1 Introduction

A globally hyperbolic spacetime is a spacetime that follows two conditions:

1. Time travel of any form is not possible.
2. Considering two arbitrary points in spacetime p, q, the intersection of the causual future and causal past of the two points must always be sequentially compact.

Let X be a (2+1)-dimensional globally hyperbolic spacetime of the form $\Sigma \times \mathbb{R}$ with Σ being a Cauchy surface that is homeomorphic to $\mathbb{R}^2$. Let N_X be the space of all light rays within X. N_x can be represented using the spherical cotangent bundle $ST^*\Sigma$ of Σ, which is homeomorphic to a solid torus. A point $p \in X$ creates a light cone that will intersect $\Sigma \times t$ in a curve parameterized by a circle. This curve defines the knot in the solid torus known as the sky S_p of p. According to the Low Conjecture, proved by Chernov and Nemirovski [2], two points $p, q \in X$ are causally related if and only if S_p and S_q are linked in N_X. Thus, link invariants that are able to distinguish whether or not S_p and S_q are linked in the torus are capable of detecting causality.

Recently, Allen and Swenberg [1] showed that the Jones polynomial likely detects causality while the Alexander-Conway polynomial likely does not. Specifically, they found a link (Figure 3) that corresponds to possibly causally related events that the Alexander-Conway polynomial could not differentiate from the link corresponding to causally unrelated events (Figure 2). Since the Alexander-Conway polynomial is likely not enough to distinguish causality, the natural question that arises is what extra information can be added to the Alexander-Conway polynomial so that it could plausibly detect causality.

In this paper, I show (using algebraic manipulation and computer computation) that the quandle coloring invariants of the Alexander quandles with $\mathbb{Z}_5$, $\mathbb{Z}_7$, and $\mathbb{Z}_{11}$ (excluding Takasaki quandles) cannot differentiate the example provided by Allen and Swenberg. This suggests that these Alexander quandles likely cannot be paired with the Alexander-Conway polynomial to plausibly detect causality.

2 Tools Utilized

2.1 Quandles

Quandles are algebraic structures (similar to groups) equipped with a set and a binary operation. However, quandles are specifically restricted so that they create a link invariant. Thus, the conditions for defining quandles are set so that they preserve the three Reidemeister moves (For more information see [5] or [3]).

Quandles are generated from a given oriented link projection by assigning a unique variable to each arc in the link. Then, the variables are related to one another through equations determined by the crossings in the link.

Crossings correspond to equations in the following manner:

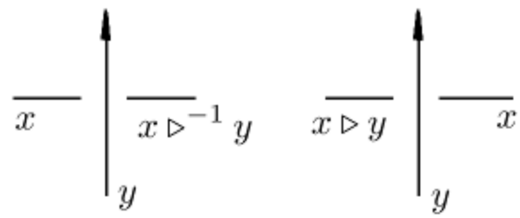
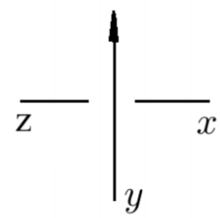


Figure 1: From [5]

For example, the following crossing would correspond to the equation $z = x \triangleright y$:

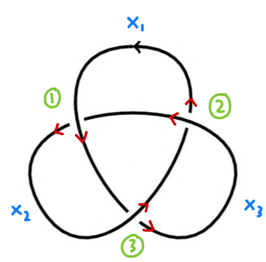


The formal definition of a quandle is a set X with a binary operation $\triangleright : X \times X \to X$ that satisfies the following conditions:

1. For all $x \in X$, $x \triangleright x = x$.
2. For all $x, y \in X$, there exists an element $z \in X$ such that $x = z \triangleright y$.
3. For all $x, y, z \in X$, $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$. (This also holds for $\triangleright^{-1}$).

(From 2. it is implied that there exists an inverse operation: $\triangleright^{-1}$, and from 3. is is implied that $\triangleright^{-1}$ is also self distributive).

By these conditions, the so called fundamental quandle (written as $Q(L)$) can be generated from any link. For example, the following is the fundamental quandle for the trefoil knot (3_1):



Crossing	Fundamental Quandle
1	$x_3 = x_2 \triangleright x_1$
2	$x_2 = x_1 \triangleright x_3$
3	$x_1 = x_3 \triangleright x_2$

$Q(3_1) = \{x_1, x_2, x_3 | x_3 = x_2 \triangleright x_1, x_2 = x_1 \triangleright x_3, x_1 = x_3 \triangleright x_2\}$

2.2 The Coloring Invariant

Definition. *A quandle homorphism is a map from one quandle to another that preserves the quandle operation. In other words, a homomorphism $f : X \to Y$ must follow the condition that $f(x \triangleright y) = f(x) \triangleright f(y)$.*

Definition. *The coloring invariant for a link is defined as the number of homomorphisms between the fundamental quandle generated from that link and a different quandle.*

In practice, the quandle coloring invariant can be computed (assuming the second quandle is in $\mathbb{Z}_n$) by finding the number of solutions to the system of equations created from the second (non-fundamental) quandle. This paper will compute coloring invariants using the Alexander quandle.

2.3 The Alexander Quandle

Let M be a module over the ring $\mathbb{Z}[t, t^{-1}]$. The Alexander quandle is defined as M under the operation $a \triangleright b = ta + (1-t)b$. (M is often set to $\mathbb{Z}_n$). Depending on what n is chosen for $\mathbb{Z}_n$ and which value is chosen for t, one can generate many different Alexander quandles.

For example, the Alexander quandle with $\mathbb{Z}_5$ and $t = 3$ for the trefoil knot shown above would be

1. $x_3 = 3x_2 + 3x_1$
2. $x_2 = 3x_1 + 3x_3$
3. $x_1 = 3x_3 + 3x_2$

3 Testing Alexander Quandle with $\mathbb{Z}_5$ and $t = 3$

In this case, $a \triangleright b = 3a + 3b$, $a, b \in \mathbb{Z}_5$.

3.1 Computation for Sum of Two Hopf Links

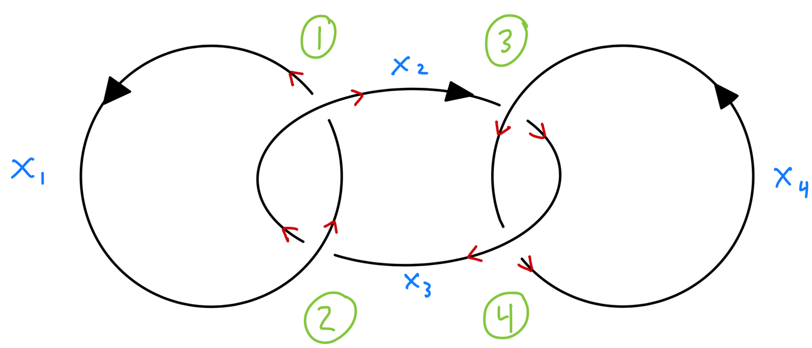


Figure 2: Labeled projection of the Sum of Two Hopf Links

I derive the following equations from Figure 2:

Crossing	Fundamental Quandle	Alexander Quandle
1	$x_1 = x_1 \triangleright x_2$	$x_1 = 3x_1 + 3x_2$
2	$x_2 = x_3 \triangleright x_1$	$x_2 = 3x_3 + 3x_1$
3	$x_3 = x_2 \triangleright x_4$	$x_3 = 3x_2 + 3x_4$
4	$x_4 = x_4 \triangleright x_3$	$x_4 = 3x_4 + 3x_3$

Table 1: System of equations corresponding to Sum of Two Hopf Links

In this case, the number of solutions to the system of equations = the number of quandle colorings invariant of the quandle. After algebraic manipulation, I solve the system of equations and arrive at the following relation:

$x_1 = x_2$
$x_1 = x_3$
$x_1 = x_4$
$x_3 = x_4$
$\Rightarrow x_1 = x_2 = x_3 = x_4$
Solutions: $(0, 0, 0, 0)$, $(1, 1, 1, 1)$, $(2, 2, 2, 2)$, $(3, 3, 3, 3)$, $(4, 4, 4, 4)$
Number of solutions: 5

3.2 Computation for Allen-Swenberg Example

First, I label the Allen-Swenberg example link (see Figure 3).
Next, I find the system of equations for the fundamental quandle and Alexander quandle of the link (see Table 2)

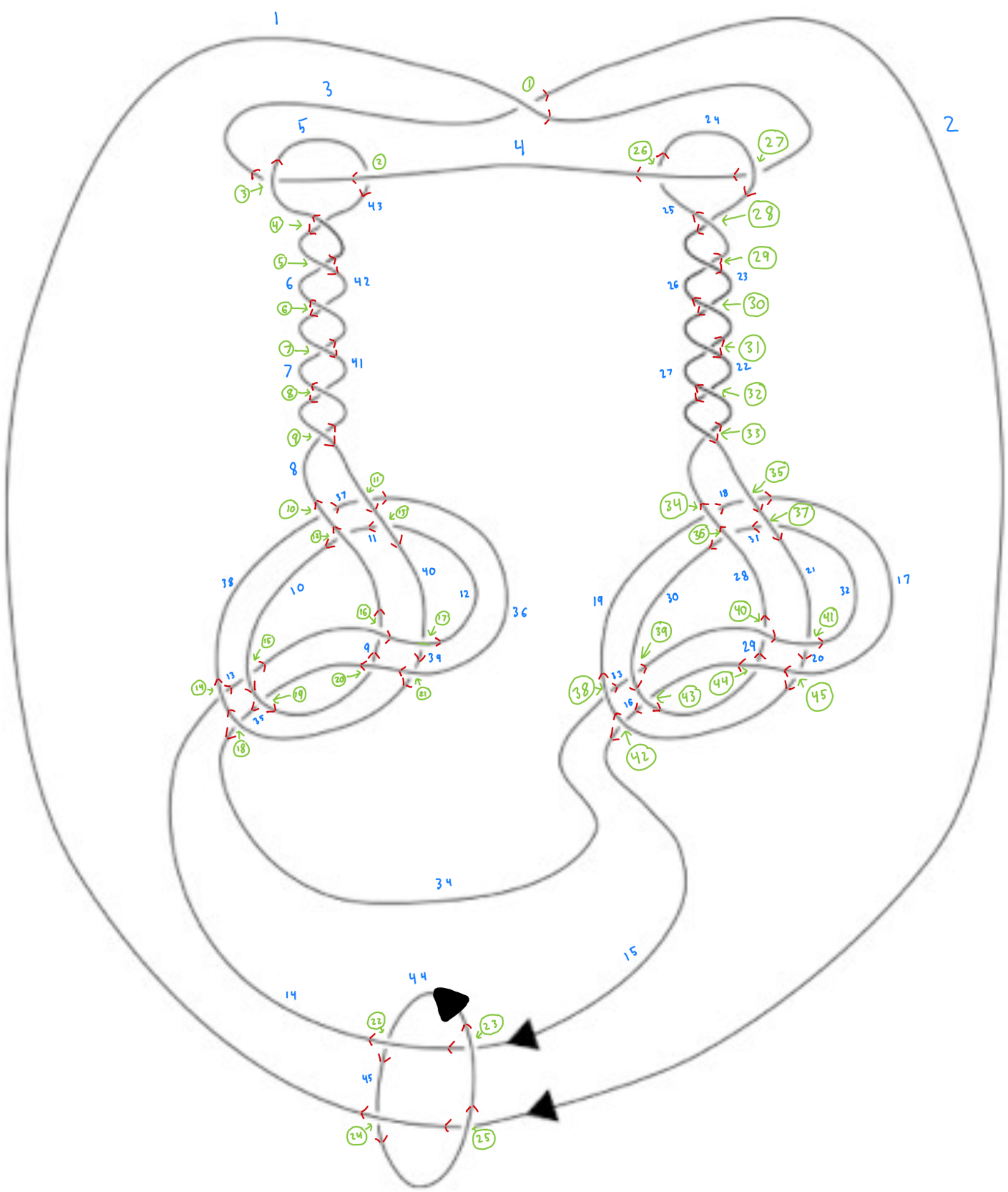


Figure 3: Labeled projection of the Allen-Swenberg example
(Arcs are labeled in blue while crossings are labeled in green)

Crossing	Fundamental Quandle	Alexander Quandle
1	$x_2 = x_3 \triangleright x_1$	$x_2 = 3x_3 + 3x_1$
2	$x_{43} = x_5 \triangleright x_4$	$x_{43} = 3x_5 + 3x_4$
3	$x_3 = x_4 \triangleright x_5$	$x_3 = 3x_4 + 3x_5$
4	$x_{42} = x_{43} \triangleright x_5$	$x_{42} = 3x_{43} + 3x_5$
5	$x_5 = x_6 \triangleright x_{42}$	$x_5 = 3x_6 + 3x_{42}$
6	$x_{41} = x_{42} \triangleright x_6$	$x_{41} = 3x_{42} + 3x_6$
7	$x_6 = x_7 \triangleright x_{41}$	$x_6 = 3x_7 + 3x_{41}$
8	$x_{40} = x_{41} \triangleright x_7$	$x_{40} = 3x_{41} + 3x_7$
9	$x_7 = x_8 \triangleright x_{40}$	$x_7 = 3x_8 + 3x_{40}$
10	$x_{38} = x_{37} \triangleright x_8$	$x_{38} = 3x_{37} + 3x_8$
11	$x_{36} = x_{37} \triangleright x_{40}$	$x_{36} = 3x_{37} + 3x_{40}$
12	$x_{10} = x_{11} \triangleright x_8$	$x_{10} = 3x_{11} + 3x_8$
13	$x_{12} = x_{11} \triangleright x_{40}$	$x_{12} = 3x_{11} + 3x_{40}$
14	$x_{14} = x_{13} \triangleright x_{38}$	$x_{14} = 3x_{13} + 3x_{38}$
15	$x_{12} = x_{13} \triangleright x_{10}$	$x_{12} = 3x_{13} + 3x_{10}$
16	$x_8 = x_9 \triangleright x_{12}$	$x_8 = 3x_9 + 3x_{12}$
17	$x_{40} = x_{39} \triangleright x_{12}$	$x_{40} = 3x_{39} + 3x_{12}$
18	$x_{34} = x_{35} \triangleright x_{38}$	$x_{34} = 3x_{35} + 3x_{38}$
19	$x_{36} = x_{35} \triangleright x_{10}$	$x_{36} = 3x_{35} + 3x_{10}$
20	$x_{10} = x_9 \triangleright x_{36}$	$x_{10} = 3x_9 + 3x_{36}$
21	$x_{38} = x_{39} \triangleright x_{36}$	$x_{38} = 3x_{39} + 3x_{36}$
22	$x_{45} = x_{44} \triangleright x_{14}$	$x_{45} = 3x_{44} + 3x_{14}$
23	$x_{14} = x_{15} \triangleright x_{44}$	$x_{14} = 3x_{15} + 3x_{44}$
24	$x_{44} = x_{45} \triangleright x_1$	$x_{44} = 3x_{45} + 3x_1$
25	$x_1 = x_2 \triangleright x_{44}$	$x_1 = 3x_2 + 3x_{44}$
26	$x_{25} = x_{24} \triangleright x_4$	$x_{25} = 3x_{24} + 3x_4$
27	$x_1 = x_4 \triangleright x_{24}$	$x_1 = 3x_4 + 3x_{24}$
28	$x_{23} = x_{24} \triangleright x_{25}$	$x_{23} = 3x_{24} + 3x_{25}$
29	$x_{25} = x_{26} \triangleright x_{23}$	$x_{25} = 3x_{26} + 3x_{23}$
30	$x_{22} = x_{23} \triangleright x_{26}$	$x_{22} = 3x_{23} + 3x_{26}$
31	$x_{26} = x_{27} \triangleright x_{22}$	$x_{26} = 3x_{27} + 3x_{22}$
32	$x_{21} = x_{22} \triangleright x_{27}$	$x_{21} = 3x_{22} + 3x_{27}$
33	$x_{27} = x_{28} \triangleright x_{21}$	$x_{27} = 3x_{28} + 3x_{21}$
34	$x_{19} = x_{18} \triangleright x_{28}$	$x_{19} = 3x_{18} + 3x_{28}$
35	$x_{17} = x_{18} \triangleright x_{21}$	$x_{17} = 3x_{18} + 3x_{21}$
36	$x_{30} = x_{31} \triangleright x_{28}$	$x_{30} = 3x_{31} + 3x_{28}$
37	$x_{32} = x_{31} \triangleright x_{21}$	$x_{32} = 3x_{31} + 3x_{21}$
38	$x_{34} = x_{33} \triangleright x_{19}$	$x_{34} = 3x_{33} + 3x_{19}$
39	$x_{32} = x_{33} \triangleright x_{30}$	$x_{32} = 3x_{33} + 3x_{30}$
40	$x_{28} = x_{29} \triangleright x_{32}$	$x_{28} = 3x_{29} + 3x_{32}$
41	$x_{21} = x_{20} \triangleright x_{32}$	$x_{21} = 3x_{20} + 3x_{32}$
42	$x_{15} = x_{16} \triangleright x_{19}$	$x_{15} = 3x_{16} + 3x_{19}$
43	$x_{17} = x_{16} \triangleright x_{30}$	$x_{17} = 3x_{16} + 3x_{30}$
44	$x_{30} = x_{29} \triangleright x_{17}$	$x_{30} = 3x_{29} + 3x_{17}$
45	$x_{19} = x_{20} \triangleright x_{17}$	$x_{19} = 3x_{20} + 3x_{17}$

Table 2: System of equations corresponding to the Allen-Swenberg example

To calculate the solutions to this system of equations, I input the system of equations into Maple [4] using the following code:

```
assign(a = 3, b = 3, c = 5)

msolve( { x_2 = a x_3 + b x_1, x_43 = a x_5 + b x_4,
x_3 = a x_4 + b x_5, x_42 = a x_43 + b x_5,
x_5 = a x_6 + b x_42, x_41 = a x_42 + b x_6,
x_6 = a x_7 + b x_41, x_40 = a x_41 + b x_7,
x_7 = a x_8 + b x_40, x_38 = a x_37 + b x_8,
x_36 = a x_37 + b x_40, x_10 = a x_11 + b x_8,
x_12 = a x_11 + b x_40, x_14 = a x_13 + b x_38,
x_12 = a x_13 + b x_10, x_8 = a x_9 + b x_12,
x_40 = a x_39 + b x_12, x_34 = a x_35 + b x_38,
x_36 = a x_35 + b x_10, x_10 = a x_9 + b x_36,
x_38 = a x_39 + b x_36, x_45 = a x_44 + b x_14,
x_14 = a x_15 + b x_44, x_44 = a x_45 + b x_1,
x_1 = a x_2 + b x_44, x_25 = a x_24 + b x_4,
x_1 = a x_4 + b x_24, x_23 = a x_24 + b x_25,
x_25 = a x_26 + b x_23, x_22 = a x_23 + b x_26,
x_26 = a x_27 + b x_22, x_21 = a x_22 + b x_27,
x_27 = a x_28 + b x_21, x_19 = a x_18 + b x_28,
x_17 = a x_18 + b x_21, x_30 = a x_31 + b x_28,
x_32 = a x_31 + b x_21, x_34 = a x_33 + b x_19,
x_32 = a x_33 + b x_30, x_28 = a x_29 + b x_32,
x_21 = a x_20 + b x_32, x_15 = a x_16 + b x_19,
x_17 = a x_16 + b x_30, x_30 = a x_29 + b x_17,
x_19 = a x_20 + b x_17}, c)
```

Maple returns the following output:

```
{x_1 = _Z1, x_10 = _Z1, x_11 = _Z1, x_12 = _Z1, x_13 = _Z1, x_14 = _Z1,
  x_15 = _Z1, x_16 = _Z1, x_17 = _Z1, x_18 = _Z1, x_19 = _Z1, x_2 = _Z1,
  x_20 = _Z1, x_21 = _Z1, x_22 = _Z1, x_23 = _Z1, x_24 = _Z1, x_25
  = _Z1, x_26 = _Z1, x_27 = _Z1, x_28 = _Z1, x_29 = _Z1, x_3 = _Z1, x_30
  = _Z1, x_31 = _Z1, x_32 = _Z1, x_33 = _Z1, x_34 = _Z1, x_35 = _Z1,
  x_36 = _Z1, x_37 = _Z1, x_38 = _Z1, x_39 = _Z1, x_4 = _Z1, x_40 = _Z1,
  x_41 = _Z1, x_42 = _Z1, x_43 = _Z1, x_44 = _Z1, x_45 = _Z1, x_5 = _Z1,
  x_6 = _Z1, x_7 = _Z1, x_8 = _Z1, x_9 = _Z1}
```

(In Maple, every value being equal to $\_Z1$ means that each value can be any number in $\mathbb{Z}_5$ with the condition that every variable is equal to one another.) Since every variable must be equal, there are 5 solutions.

3.3 Comparing Invariants

The coloring invariant for the Sum of Two Hopf Links is 5, while the coloring invariant for the Allen-Swenberg example is also 5. Since $5 = 5$, the Alexander quandle with $\mathbb{Z}_5$ and $t = 3$ does not distinguish the two links.

4 Testing Other Alexander Quandles

Using the same methods shown in section 3, I compute the coloring invariants for several Alexander quandles shown in the table below:

Alexander Quandle	Operation	Solutions for Link 1	Solutions for Link 2	Distinguishes?
$\mathbb{Z}_5, t = 2$	$a \triangleright b = 2a + 4b$	5	5	No
$\mathbb{Z}_5, t = 4$	$a \triangleright b = 4a + 2b$	5	5	No
$\mathbb{Z}_7, t = 2$	$a \triangleright b = 2a + 6b$	7	7	No
$\mathbb{Z}_7, t = 3$	$a \triangleright b = 3a + 5b$	7	7	No
$\mathbb{Z}_7, t = 4$	$a \triangleright b = 4a + 4b$	7	7	No
$\mathbb{Z}_7, t = 5$	$a \triangleright b = 5a + 3b$	7	7	No
$\mathbb{Z}_11, t = 2$	$a \triangleright b = 2a + 10b$	11	11	No
$\mathbb{Z}_11, t = 3$	$a \triangleright b = 3a + 9b$	11	11	No
$\mathbb{Z}_11, t = 4$	$a \triangleright b = 4a + 8b$	11	11	No
$\mathbb{Z}_11, t = 5$	$a \triangleright b = 5a + 7b$	11	11	No
$\mathbb{Z}_11, t = 6$	$a \triangleright b = 6a + 6b$	11	11	No
$\mathbb{Z}_11, t = 7$	$a \triangleright b = 7a + 5b$	11	11	No
$\mathbb{Z}_11, t = 8$	$a \triangleright b = 8a + 4b$	11	11	No
$\mathbb{Z}_11, t = 9$	$a \triangleright b = 9a + 3b$	11	11	No

Table 3: (Link 1 is the Sum of Two Hopf Links, while Link 2 is the Allen-Swenberg example).

Remark. *Note that the number of solutions for each quandle in the table above equals the number n for the $\mathbb{Z}_n$ of that quandle. In this case, each set of solutions corresponds to this number n, because each represents the set of solutions where all values are equal. These types of solutions are specifically called the "trivial" solutions. This is because it can be proven that all links must have these solutions. Interestingly, none of the quandle invariants for the tested examples above have solutions other than the trivial solutions. As a result of this, none of the quandles can differentiate the two links.*

Conclusion

The results of this paper seem to indicate that the Alexander quandle does not contain enough information to detect causality when paired with the Alexander-Conway polynomial. Based on the evidence so far, I suspect that Alexander quandles with $n > 11$ (for $\mathbb{Z}_n$) will also not be able to differentiate the Sum of Two Hopf Links from the Allen-Swenberg example.

Acknowledgement.

This project was completed in Summer 2021 as a part of the Horizon Academic Research Program organized by Sunrise International Education Inc. This project was created by Vladimir Chernov and the program was supervised by Vladimir Chernov and Emanuele Zappala.

References

[1] Samantha Allen and Jacob H Swenberg. "Do link polynomials detect causality in globally hyperbolic spacetimes?" In: *Journal of Mathematical Physics* 62.3 (2021), p. 032503.

[2] Vladimir Chernov and Stefan Nemirovski. "Legendrian links, causality, and the Low conjecture". In: *Geometric and Functional Analysis* 19.5 (2010), pp. 1320–1333.

[3] David Joyce. "A classifying invariant of knots, the knot quandle". In: *Journal of Pure and Applied Algebra* 23.1 (1982), pp. 37–65.

[4] Maplesoft, a division of Waterloo Maple Inc.. *Maple*. Version 2021. Waterloo, Ontario, 2021. URL: https://www.maplesoft.com.

[5] Sam Nelson. *Quandles and Racks*. 2015. URL: https://www1.cmc.edu/pages/faculty/VNelson/quandles.html.