Quandles in Causality

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August 2021

Abstract

This paper aims to study the colorability of links proposed by Samantha Allen and Jacob H. Swenberg using the Alexander quandle. There are two examples of links which are important for the study of causality; Allen-Swenberg's paper shows that Jones Polynomial is most likely enough to distinguish causality in 2+1 dimensional globally hyperbolic spacetimes, but Alexander-Conway Polynomial is likely not enough. I have checked what extra information would plausibly be needed along with Alexander-Conway Polynomial to distinguish causality. I have used the Alexander Quandle by considering the set, $\mathbb{Z}_n$, integers modulo n, and an integer t co-prime to n, n being the element of the set of prime numbers. I have gotten that it does not distinguish the Allen-Swenberg examples, so it can not help distinguish causality.

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1 Introduction

Let X be a 2+1 dimensional globally hyperbolic spacetime with Cauchy surface Σ homeomorphic to $\mathbb{R}^2$, and let N be the set of future-directed null geodesics in X. The set N can be identified with the spherical cotangent bundle $ST^*\Sigma$ of Σ, which in this case is homeomorphic to a solid torus $S^1 \times \mathbb{R}^2$. The sky of $x \in X$, denoted $S_x \subset N$, is the set of all future-directed null geodesics through x. The sky S_x is homeomorphic to a circle, and viewed as a subset of the solid torus $S^1 \times \mathbb{R}^2$, S_x is isotopic to $S^1 \times \{0\}$. For more explanations, see [1] or [2].

Nemirovski and Chernov [3] proved the Low conjecture, which says that as long as Σ is not a closed 2-manifold, two events $x, y \in X$ are causally related if and only if their skies $S_x \sqcup S_y$ are linked. In this context (Σ homeomorphic to $\mathbb{R}^2$), *linked* means either $S_x \cap S_y \neq \emptyset$, or $S_x \sqcup S_y$ is not isotopic in N to $S^1 \times \{a\} \sqcup S^1 \times \{b\}$ for $a, b \in \mathbb{R}^2$, and causally related means one of the events can influence other event. Nemirovski and Chernov actually proved more; they showed that the relationship between linking and causality holds as long as Σ is not homeomorphic to S^2 or $\mathbb{R}P^2$. They also proved the Legendrian Low conjecture, which is analogous to the Low conjecture but for higher-dimensional spacetimes, where topological linking is replaced by Legendrian linking.

A natural question is whether linking of S_x and S_y, and thus causality, could be detected by various link invariants. Natàrio and Tod [2] provided a large family of pairs of skies corresponding to causally related events such that, for each pair, the associated link has nontrivial Kauffman polynomial. The Kauffman polynomial is related to the Jones polynomial $V(L)$ by a change of variables, but there is another invariant that contains strictly more information than both. Specifically, Khovanov homology provides a "categorification" of the Jones polynomial [4]. In fact, Khovanov homology detects causality in X [1]. Another common link polynomial is the Alexander-Conway polynomial (also called the Conway polynomial) $\nabla(L)$, which is categorified by Heegaard Floer homology [5]. Chernov, Martin, and Petkova have proved that link Heegaard Floer homology will also detect causality in this setting [2].The related knot polynomials, the Jones polynomial and the Conway polynomial, are strictly weaker link invariants than their respective categorifications. On the other hand, the work of Natàrio and Tod indicates that these polynomials still might detect causality.

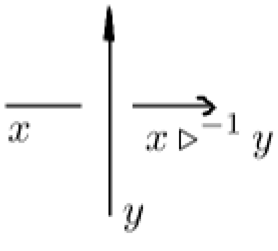
In this paper, I check with the calculation that whether Alexander Quandle is able to distinguish the two examples of Allen-Swenberg [6] or not.

2 What is a Quandle?

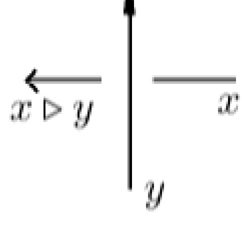
A quandle is a set X with a binary operation that satisfies

1. $x \triangleright x = x$ for all elements $x \in X$,
2. For every pair of elements $x, y \in X$, there is a unique element $z \in X$ such that $x = y \triangleright z$, and
3. $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$ for all $x, y, z \in X$.

Axiom 2 is equivalent to the quandle operation having a *right inverse*, that is, a second operation $\triangleright^{-1}$ such that $(x \triangleright y) \triangleright^{-1} y = x$ for all $x, y \in X$.
This quandle operation is generally non-commutative and non-associative, i.e., in general $x \triangleright y \neq y \triangleright x$ and $(x \triangleright y) \triangleright z \neq x \triangleright (y \triangleright z)$.
The relationship between quandles and knots was established by David Joyce in 1982, where the knot quandle is defined. Specifically, I take a knot diagram and assign a letter to each arc in the diagram, i.e. the quandle elements are labels for the arcs. Then at each crossing, I have pictured the relationship between the arcs shown below.



(a) Negative crossing

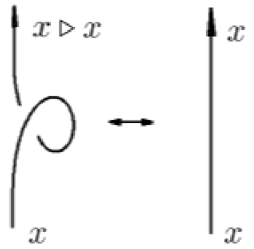


(b) Positive crossing

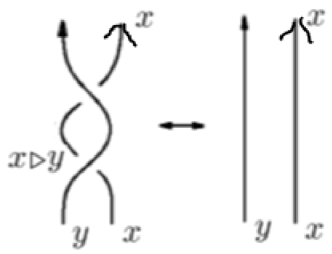
Figure 1: Quandle operation at crossings. [Source: [8]]

The arc labeled $x \triangleright^{-1} y$ is the result of the arc labeled x crossing under the arc labeled y from left to right, while the arc labeled $x \triangleright y$ is the result of the arc labeled x crossing under the arc labeled y from right to left. To check whether Quandle is an invariant of knot, I need to verify that Reidemeister moves don't change the quandle. The quandle axioms are just the conditions required for all the labels on the edges of the diagrams to be identical before and after the Reidemeister moves. These pictures are meant to represent two portions of knot diagrams, and the portions of the diagrams outside the pictured parts are identical. In particular, the labels on the arcs at the edges of the pictures have to be the same before and after the move.

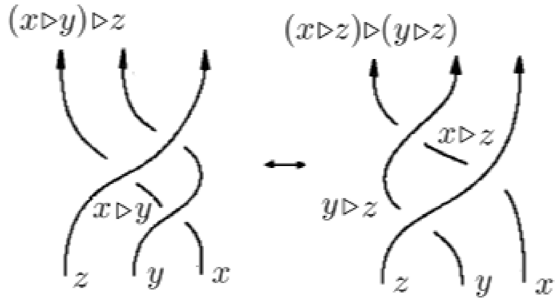
The Reidemeister moves translate into the quandle axioms, namely:



(a) 1st axiom relates to Reidemeister Move I



(b) 2nd axiom relates Reidemeister Move II



(c) 3rd axiom relates Reidemeister Move III

Figure 2: Quandle operation in Reidemeister moves. [Source : [8]]

Thus, if the labels on the arcs come from a quandle and are chosen according to the crossing rule above, then for every labeling of the diagram before the move, there is exactly one corresponding labeling by the same quandle after the move. That is, the total number of labeling of a knot diagram by elements of a fixed finite quandle satisfying the labeling condition is the same for any two diagrams of the knot. Hence, to prove that two knot diagrams represent distinct knots, we can count the number of labeling of the two diagrams by the same finite quandle which satisfy the labeling condition. If the numbers are the same, then the test doesn't tell us anything, but if the numbers are different, then the diagrams must represent different knots. In this way, I get a knot invariant from a finite quandle, known as the counting invariant, denoted by$|$ $Hom(Q(K), Q)$ $|$ where K is any knot diagram.

3 Alexander Quandle

An example of a quandle structure is Alexander quandle. It can be constructed by considering the set $\mathbb{Z}_n$ of integers modulo n and choosing an integer t which is co-prime to n. The Alexander quandle, $\Lambda_{n,t}$, can then be obtained by the underlying set $\mathbb{Z}_n$ by defining the binary operation as

$$x \triangleright y = t\,x + (1-1)\,y$$

It is easy to see this forms a quandle:

1. For all $x \in X$, $x \triangleright x = t\,x + (1-t)\,x = x$.
2. For every pair of elements $x, y \in X$, there is a unique element $z \in X$ such that $x = y \triangleright z = t\,x + (1-t)\,y$ and $(x \triangleright y) \triangleright^{-1} y = x$.
3. Similarly, a straightforward computation verifies that $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$.

For example, if one chooses $n = 3$ and $t = 2$, then $x \triangleright y = 2x + (1-2)y = 2x - y = 2x + 2y \; mod\, 3$. To show how this quandle operation has been applied in knots considering the set $\mathbb{Z}_3$, I have taken an oriented Trefoil knot, shown in Figure 3. I have labeled the crossing number along with the arcs.

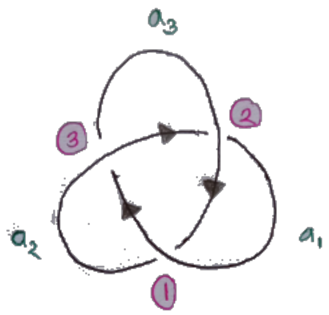


Figure 3: Trefoil knot

The table shows the results after applying the Alexander quandle operation in Trefoil knot :

Crossing number	Equations at each crossing
1	$a_2 = a_3 \triangleright a_1 \Rightarrow a_2 = 2a_3 - a_1 \Rightarrow a_2 + a_3 + a_1 = 0 \; mod\, 3$
2	$a_1 = a_2 \triangleright a_3 \Rightarrow a_1 = 2a_2 - a_3 \Rightarrow a_1 + a_2 + a_3 = 0 \; mod\, 3$
3	$a_3 = a_1 \triangleright a_2 \Rightarrow a_3 = 2a_1 - a_2 \Rightarrow a_3 + a_1 + a_2 = 0 \; mod\, 3$

Table 1: Alexander Quandle computation in Trefoil

The equation at each crossing of Trefoil knot is equal. Hence, the number of labeling has to be calculated from this equation, $a_1 + a_2 + a_3 = 0 \; mod\, 3$, by direct computation, i.e.

$$0 + 0 + 0 = 0\, mod\, 3$$

$$1 + 1 + 1 = 0\, mod\, 3$$

$$2 + 2 + 2 = 0\, mod\, 3$$

$$1 + 2 + 0 = 0\, mod\, 3$$

$$2 + 1 + 0 = 0\, mod\, 3$$

$$2 + 0 + 1 = 0\, mod\, 3$$

$$0 + 2 + 1 = 0\, mod\, 3$$

$$0 + 1 + 2 = 0\, mod\, 3$$

$$1 + 0 + 2 = 0\, mod\, 3$$

$\therefore$ The number of labeling for Trefoil knot in $\Lambda_{3,2}$ is 9,

i.e. $\{a_1, a_2, a_3\} = \{0,0,0\}, \{1,1,1\}, \{2,2,2\}, \{1,2,0\}, \{2,1,0\}, \{2,0,1\}, \{0,2,1\}, \{0,1,2\}, \{1,0,2\}\, mod\, 3$

4 Distinguishing causality by Alexander Quandles

In the study of causality, the two examples of Allen-Swenberg paper are important : Figure 4 and Figure 5; both of the link diagrams have to be oriented. After applying the fixed finite Alexander Quandle operation in both of the links, I compare the results of them. If the numbers of labeling in both diagrams are the same, then it can not be concluded that both of the examples of Allen-Swenberg paper are the same link or distinct links, but if the numbers of labeling are different, then the both of the examples indeed represent different links.

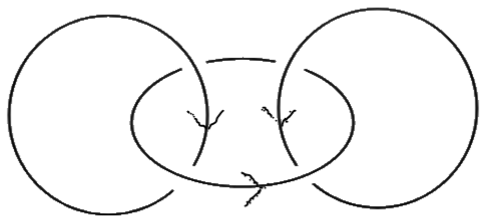


Figure 4: Connected sum of two Hopf links. [Source: [6]]

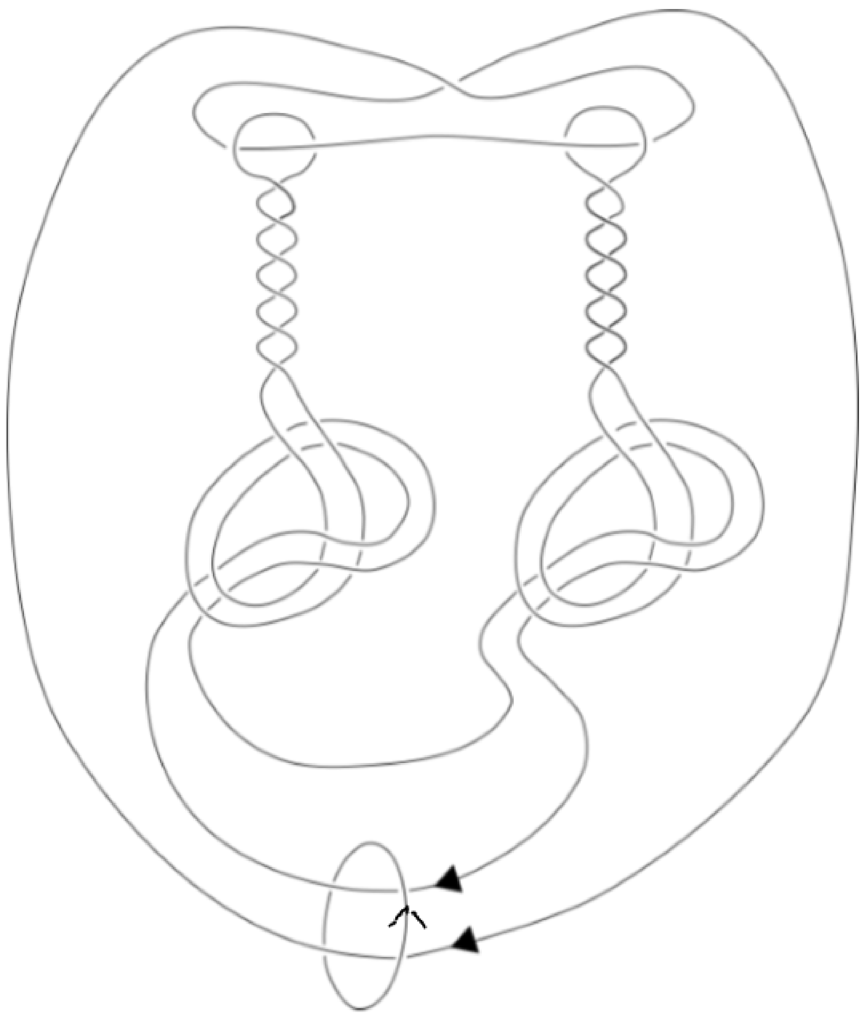


Figure 5: Another example of Allen-Swenberg. [Source: [6]]

In order to calculate the number of labeling of the connected sum of two Hopf links and Allen-Swenberg example in $\Lambda_{5,3}$, I label the crossing numbers and the arcs, as shown in Figure 6 and Figure 7, then apply the Alexander quandle operation.

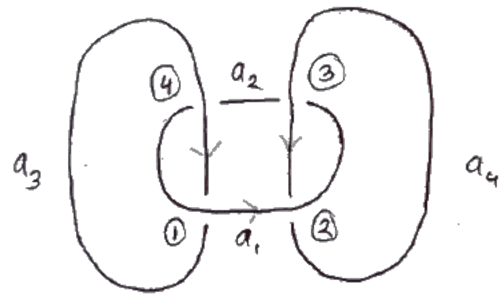


Figure 6: Connected Sum of Two oriented Hopf links

The table shows the results after applying the Alexander quandle operation in the connected sum of two oriented Hopf Links :

Crossing number	Equations at each crossing
1	$a_3 = a_3 \triangleright a_1 \Rightarrow a_3 = 3a_3 - 2a_1 \Rightarrow 2a_3 - 2a_1 = 0 \Rightarrow a_1 = a_3 mod\, 5$
2	$a_4 = a_4 \triangleright a_1 \Rightarrow a_4 = 3a_4 - 2a_1 \Rightarrow 2a_4 - 2a_1 = 0 \Rightarrow a_1 = a_4 mod\, 5$
3	$a_1 = a_2 \triangleright a_4 \Rightarrow a_1 = 3a_2 - 2a_4 \Rightarrow a_1 + 2a_2 + 2a_4 = 0\ mod\, 5$
4	$a_2 = a_1 \triangleright a_3 \Rightarrow a_2 = 3a_1 - 2a_3 \Rightarrow a_2 + 2a_1 + 2a_3 = 0\ mod\, 5$

Table 2: Alexander quandle computation in the connected sum of two oriented Hopf links

Solving four of these equations,

From crossing number 1, $a_1 = a_3$
From crossing number 2, $a_1 = a_4$
From crossing number 3 and 4, $3a_1 + 2a_2 = 0\ mod\, 5$

Hence, the number of labeling has to be calculated from this equation, $3a_1 + 2a_2 = 0\ mod\, 5$, by direct computation, i.e.

$$3(0) + 2(0) = 0 \Rightarrow 0 + 0 = 0\, mod\, 5$$

$$3(1) + 2(1) = 0 \Rightarrow 3 + 2 = 0\, mod\, 5$$

$$3(2) + 2(2) = 0 \Rightarrow 6 + 4 = 0\, mod\, 5$$

$$3(3) + 2(3) = 0 \Rightarrow 9 + 6 = 0\, mod\, 5$$

$$3(4) + 2(4) = 0 \Rightarrow 12 + 8 = 0\, mod\, 5$$

$\therefore$ The number of labeling for the connected sum of two oriented Hopf links in $\Lambda_{5,3}$ is 5,

i.e. $\{a_1, a_2, a_3, a_4\} = \{0,0,0,0\}, \{1,1,1,1\}, \{2,2,2,2\}, \{3,3,3,3\}, \{4,4,4,4\}\, mod\, 5$

Figure 7 shows the Allen-Swenberg example after giving orientation, labeling the crossing numbers and arcs :

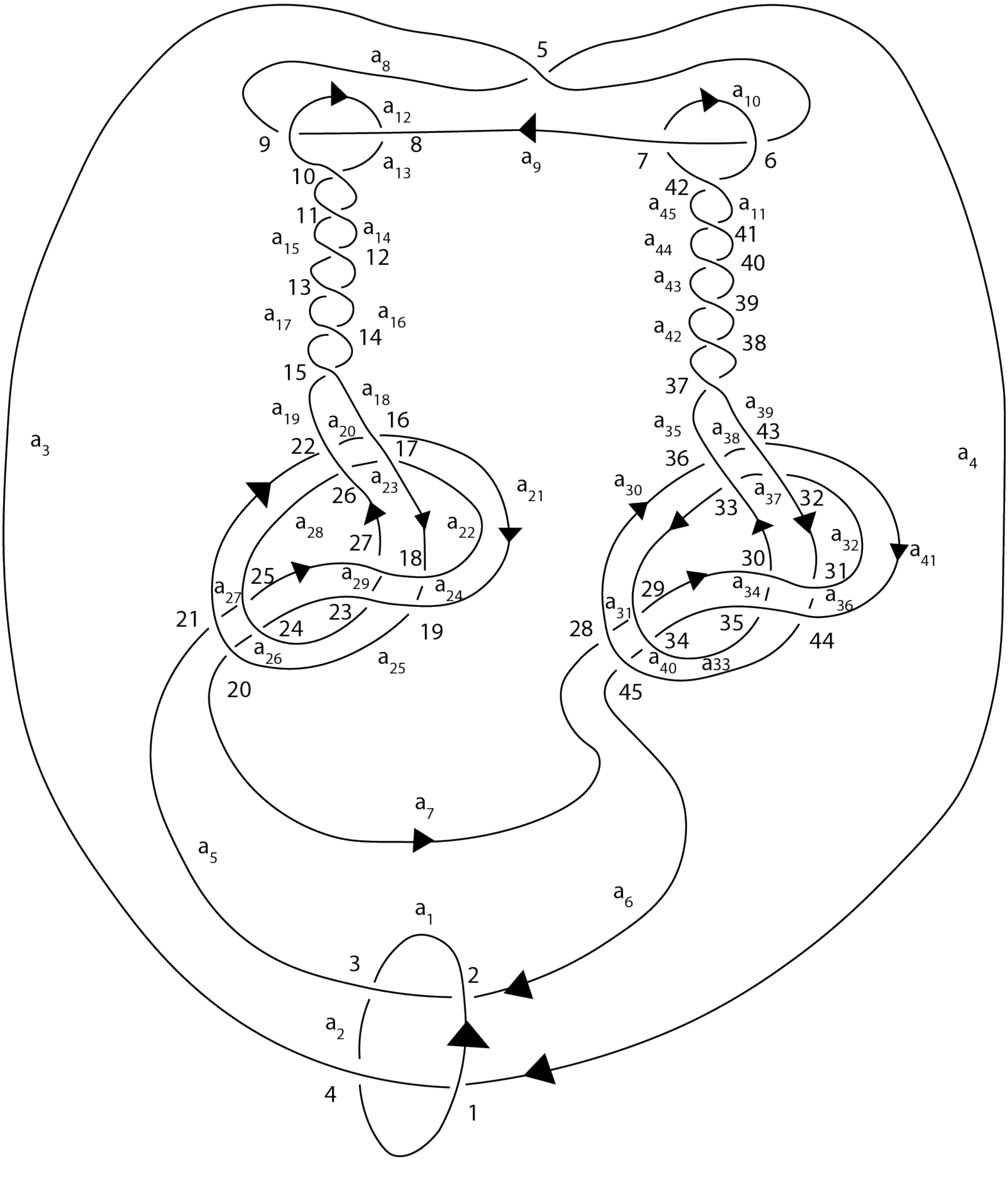


Figure 7: Allen-Swenberg example

The table shows the results after applying the Alexander quandle operation in the Allen-Swenberg example :

Crossing Number	Equations at each crossing
1	$a_3 = a_4 \triangleright a_1 \Rightarrow a_3 = 3a_4 - 2a_1 \Rightarrow a_3 + 2a_4 + 2a_1 = 0\ mod\, 5$
2	$a_5 = a_6 \triangleright a_1 \Rightarrow a_5 = 3a_6 - 2a_1 \Rightarrow a_5 + 2a_6 + 2a_1 = 0\ mod\, 5$
3	$a_2 = a_1 \triangleright a_5 \Rightarrow a_2 = 3a_1 - 2a_5 \Rightarrow a_2 + 2a_1 + 2a_5 = 0\ mod\, 5$
4	$a_1 = a_2 \triangleright a_3 \Rightarrow a_1 = 3a_2 - 2a_3 \Rightarrow a_1 + 2a_2 + 2a_3 = 0\ mod\, 5$
5	$a_4 = a_8 \triangleright a_3 \Rightarrow a_4 = 3a_8 - 2a_3 \Rightarrow a_4 + 2a_8 + 2a_3 = 0\ mod\, 5$
6	$a_3 = a_9 \triangleright a_{10} \Rightarrow a_3 = 3a_9 - 2a_{10} \Rightarrow a_3 + 2a_9 + 2a_{10} = 0\ mod\, 5$
7	$a_{11} = a_{10} \triangleright a_9 \Rightarrow a_{11} = 3a_{10} - 2a_9 \Rightarrow a_{11} + 2a_{10} + 2a_9 = 0\ mod\, 5$
8	$a_{13} = a_{12} \triangleright a_9 \Rightarrow a_{13} = 3a_{12} - 2a_9 \Rightarrow a_{13} + 2a_{12} + 2a_9 = 0\ mod\, 5$
9	$a_8 = a_9 \triangleright a_{12} \Rightarrow a_8 = 3a_9 - 2a_{12} \Rightarrow a_8 + 2a_9 + 2a_{12} = 0\ mod\, 5$
10	$a_{14} = a_{13} \triangleright a_{12} \Rightarrow a_{14} = 3a_{13} - 2a_{12} \Rightarrow a_{14} + 2a_{13} + 2a_{12} = 0\ mod\, 5$
11	$a_{12} = a_{15} \triangleright a_{14} \Rightarrow a_{12} = 3a_{15} - 2a_{14} \Rightarrow a_{12} + 2a_{15} + 2a_{14} = 0\ mod\, 5$
12	$a_{16} = a_{14} \triangleright a_{15} \Rightarrow a_{16} = 3a_{14} - 2a_{15} \Rightarrow a_{16} + 2a_{14} + 2a_{15} = 0\ mod\, 5$
13	$a_{15} = a_{17} \triangleright a_{16} \Rightarrow a_{15} = 3a_{17} - 2a_{16} \Rightarrow a_{15} + 2a_{17} + 2a_{16} = 0\ mod\, 5$
14	$a_{18} = a_{16} \triangleright a_{17} \Rightarrow a_{18} = 3a_{16} - 2a_{17} \Rightarrow a_{18} + 2a_{16} + 2a_{17} = 0\ mod\, 5$
15	$a_{17} = a_{19} \triangleright a_{18} \Rightarrow a_{17} = 3a_{19} - 2a_{18} \Rightarrow a_{17} + 2a_{19} + 2a_{18} = 0\ mod\, 5$
16	$a_{21} = a_{20} \triangleright a_{18} \Rightarrow a_{21} = 3a_{20} - 2a_{18} \Rightarrow a_{21} + 2a_{20} + 2a_{18} = 0\ mod\, 5$
17	$a_{22} = a_{23} \triangleright a_{18} \Rightarrow a_{22} = 3a_{23} - 2a_{18} \Rightarrow a_{22} + 2a_{23} + 2a_{18} = 0\ mod\, 5$
18	$a_{18} = a_{24} \triangleright a_{22} \Rightarrow a_{18} = 3a_{24} - 2a_{22} \Rightarrow a_{18} + 2a_{24} + 2a_{22} = 0\ mod\, 5$
19	$a_{25} = a_{24} \triangleright a_{21} \Rightarrow a_{25} = 3a_{24} - 2a_{21} \Rightarrow a_{25} + 2a_{24} + 2a_{21} = 0\ mod\, 5$
20	$a_7 = a_{26} \triangleright a_{25} \Rightarrow a_7 = 3a_{26} - 2a_{25} \Rightarrow a_7 + 2a_{26} + 2a_{25} = 0\ mod\, 5$
21	$a_5 = a_{27} \triangleright a_{25} \Rightarrow a_5 = 3a_{27} - 2a_{25} \Rightarrow a_5 + 2a_{27} + 2a_{25} = 0\ mod\, 5$
22	$a_{25} = a_{20} \triangleright a_{19} \Rightarrow a_{25} = 3a_{20} - 2a_{19} \Rightarrow a_{25} + 2a_{20} + 2a_{19} = 0\ mod\, 5$
23	$a_{28} = a_{29} \triangleright a_{21} \Rightarrow a_{28} = 3a_{29} - 2a_{21} \Rightarrow a_{28} + 2a_{29} + 2a_{21} = 0\ mod\, 5$
24	$a_{21} = a_{26} \triangleright a_{28} \Rightarrow a_{21} = 3a_{26} - 2a_{28} \Rightarrow a_{21} + 2a_{26} + 2a_{28} = 0\ mod\, 5$
25	$a_{22} = a_{27} \triangleright a_{28} \Rightarrow a_{22} = 3a_{27} - 2a_{28} \Rightarrow a_{22} + 2a_{27} + 2a_{28} = 0\ mod\, 5$
26	$a_{28} = a_{23} \triangleright a_{19} \Rightarrow a_{28} = 3a_{23} - 2a_{19} \Rightarrow a_{28} + 2a_{23} + 2a_{19} = 0\ mod\, 5$
27	$a_{19} = a_{29} \triangleright a_{22} \Rightarrow a_{19} = 3a_{20} - 2a_{22} \Rightarrow a_{19} + 2a_{20} + 2a_{22} = 0\ mod\, 5$
28	$a_7 = a_{31} \triangleright a_{30} \Rightarrow a_7 = 3a_{31} - 2a_{30} \Rightarrow a_7 + 2a_{31} + 2a_{30} = 0\ mod\, 5$
29	$a_{32} = a_{31} \triangleright a_{33} \Rightarrow a_{32} = 3a_{31} - 2a_{33} \Rightarrow a_{32} + 2a_{31} + 2a_{33} = 0\ mod\, 5$
30	$a_{35} = a_{34} \triangleright a_{32} \Rightarrow a_{35} = 3a_{34} - 2a_{32} \Rightarrow a_{35} + 2a_{34} + 2a_{32} = 0\ mod\, 5$
31	$a_{39} = a_{36} \triangleright a_{32} \Rightarrow a_{39} = 3a_{36} - 2a_{32} \Rightarrow a_{39} + 2a_{36} + 2a_{32} = 0\ mod\, 5$
32	$a_{32} = a_{37} \triangleright a_{39} \Rightarrow a_{32} = 3a_{37} - 2a_{39} \Rightarrow a_{32} + 2a_{37} + 2a_{39} = 0\ mod\, 5$
33	$a_{33} = a_{37} \triangleright a_{35} \Rightarrow a_{33} = 3a_{37} - 2a_{35} \Rightarrow a_{33} + 2a_{37} + 2a_{35} = 0\ mod\, 5$
34	$a_{41} = a_{40} \triangleright a_{33} \Rightarrow a_{41} = 3a_{40} - 2a_{33} \Rightarrow a_{41} + 2a_{40} + 2a_{33} = 0\ mod\, 5$
35	$a_{33} = a_{34} \triangleright a_{41} \Rightarrow a_{33} = 3a_{34} - 2a_{41} \Rightarrow a_{33} + 2a_{34} + 2a_{41} = 0\ mod\, 5$
36	$a_{30} = a_{38} \triangleright a_{35} \Rightarrow a_{30} = 3a_{38} - 2a_{35} \Rightarrow a_{30} + 2a_{38} + 2a_{35} = 0\ mod\, 5$
37	$a_{42} = a_{35} \triangleright a_{39} \Rightarrow a_{42} = 3a_{35} - 2a_{39} \Rightarrow a_{42} + 2a_{35} + 2a_{39} = 0\ mod\, 5$
38	$a_{39} = a_{43} \triangleright a_{42} \Rightarrow a_{39} = 3a_{43} - 2a_{42} \Rightarrow a_{39} + 2a_{43} + 2a_{42} = 0\ mod\, 5$
39	$a_{44} = a_{42} \triangleright a_{43} \Rightarrow a_{44} = 3a_{42} - 2a_{43} \Rightarrow a_{44} + 2a_{42} + 2a_{43} = 0\ mod\, 5$
40	$a_{43} = a_{45} \triangleright a_{44} \Rightarrow a_{43} = 3a_{45} - 2a_{44} \Rightarrow a_{43} + 2a_{45} + 2a_{44} = 0\ mod\, 5$
41	$a_{11} = a_{44} \triangleright a_{45} \Rightarrow a_{11} = 3a_{44} - 2a_{45} \Rightarrow a_{11} + 2a_{44} + 2a_{45} = 0\ mod\, 5$
42	$a_{45} = a_{10} \triangleright a_{11} \Rightarrow a_{45} = 3a_{10} - 2a_{11} \Rightarrow a_{45} + 2a_{10} + 2a_{11} = 0\ mod\, 5$
43	$a_{41} = a_{38} \triangleright a_{39} \Rightarrow a_{41} = 3a_{38} - 2a_{39} \Rightarrow a_{41} + 2a_{38} + 2a_{39} = 0\ mod\, 5$
44	$a_{30} = a_{36} \triangleright a_{41} \Rightarrow a_{30} = 3a_{36} - 2a_{41} \Rightarrow a_{30} + 2a_{36} + 2a_{41} = 0\ mod\, 5$
45	$a_6 = a_{40} \triangleright a_{30} \Rightarrow a_6 = 3a_{40} - 2a_{30} \Rightarrow a_6 + 2a_{40} + 2a_{30} = 0\ mod\, 5$

Table 3: Alexander quandle computation in the Allen-Swenberg example

Solving this set of equations is extremely tedious, so the use of computer computation is needed here. Wolfram Mathematica is helpful in this case. I have used the syntax below in order to solve the equations.

$$\begin{gathered} Solve\,[a3+2a4+2a1==0, a5+2a6+2a1==0, a2+2a1+2a5==0, a1+2a2+2a3==0, \\ a4+2a8+2a3==0, a3+2a9+2a10==0, a11+2a10+2a9==0, a13+2a12+2a9==0, \\ a8+2a9+2a12==0, a14+2a13+2a12==0, a12+2a15+2a14==0, a16+2a14+2a15==0, \\ a15+2a17+2a16==0, a18+2a16+2a17==0, a17+2a19+2a18==0, a21+2a20+2a18==0, \\ a22+2a23+2a18==0, a18+2a24+2a22==0, a25+2a24+2a21==0, a7+2a26+2a25==0, \\ a5+2a27+2a25==0, a25+2a20+2a19==0, a28+2a29+2a21==0, a21+2a26+2a28==0, \\ a22+2a27+2a28==0, a28+2a23+2a19==0, a19+2a29+2a22==0, a7+2a31+2a30==0, \\ a32+2a31+2a33==0, a35+2a34+2a32==0, a39+2a36+2a32==0, a32+2a37+2a39==0, \\ a33+2a37+2a35==0, a41+2a40+2a33==0, a33+2a34+2a41==0, a30+2a38+2a35==0, \\ a42+2a35+2a39==0, a39+2a43+2a42==0, a44+2a42+2a43==0, a43+2a45+2a44==0, \\ a11+2a44+2a45==0, a45+2a10+2a11==0, a41+2a38+2a39==0, a30+2a36+2a41==0, \\ a6+2a40+2a30==0\}, \{a1, a2, a3, a4, a5, a6, a7, a8, a9, a10, a11, a12, a13, a14, a15, a16, a17, \\ a18, a19, a20, a21, a22, a23, a24, a25, a26, a27, a28, a29, a30, a31, a32, a33, a34, a35, a36, a37, \\ a38, a39, a40, a41, a42, a43, a44, a45\}, Modulus \to 5] \end{gathered}$$

The system shows [1] that all the variables are equal to each other, i.e.
$a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = a_8 = a_9 = a_{10} = a_{11} = a_{12} = a_{13} = a_{14} = a_{15}$
$= a_{16} = a_{17} = a_{18} = a_{19} = a_{20} = a_{21} = a_{22} = a_{23} = a_{24} = a_{25} = a_{26} = a_{27} = a_{28} = a_{29} = a_{30}$
$= a_{31} = a_{32} = a_{33} = a_{34} = a_{35} = a_{36} = a_{37} = a_{38} = a_{39} = a_{40} = a_{41} = a_{42} = a_{43} = a_{44} = a_{45}$

$\therefore$ The number of labeling for the Allen-Swenberg example in $\Lambda_{5,3}$ is 5, i.e. all of the variables are equal to 0 or 1 or 2 or 3 or 4.

5 Conclusion

As the number of quandle counting invariant for the connected sum of two Hopf links and Allen-Swenberg example is equal, therefore this invariant does not allow one to distinguish these two links. Therefore, it is certain that this particular Alexander quandle even together with Alexander-Conway Polynomial may not be used to help capture causality.

Acknowledgments

This project was completed in Summer 2021 as a part of the Horizon Academic Research Program organized by the Sunrise International Education Inc.The program was supervised by Vladimir Chernov of Dartmouth College and Emanuele Zappala of Yale University. I would like to thank Professor Vladimir Chernov and Emanuele Zappala for the huge support as research mentors.

[1] The screenshot of the computation result of the connected sum of two Hopf links and the Allen-Swenberg example in Wolfram Mathematica is shown in Appndix A and B, respectively.

Appendix A

```
In[7]:= Solve[{2 a3 - 2 a1 == 0, 2 a4 - 2 a1 == 0,  a1 + 2 a2 + 2 a4 == 0, a2 + 2 a1 + 2 a3 == 0}, {a1, a2, a3, a4}, Modulus → 5]
        ··· Solve : Equations may not give solutions for all "solve" variables.
Out[7]= {{a4 → a1, a3 → a1, a2 → a1}}
```

Figure 8: Computation results of the connected sum of two Hopf links in Wolfram Mathematica.

Appendix B

```
In[8]:= Solve[ { a3 + 2 a4 + 2 a1 == 0, a5 +2 a6 + 2 a1 == 0, a2 + 2 a1 + 2 a5 == 0, a1 + 2 a2 + 2 a3 == 0, a4 + 2 a8 + 2 a3 == 0, a3 + 2 a9 + 2 a10 == 0, a11 + 2 a10 + 2 a9 == 0, a13 + 2
     a12 + 2 a9 == 0, a8 + 2 a9 + 2 a12 == 0, a14 + 2 a13 + 2 a12 == 0, a12 + 2 a15 + 2 a14 == 0, a16 + 2 a14 + 2 a15 == 0, a15 + 2 a17 + 2 a16 == 0, a18 + 2 a16 + 2 a17 == 0,
     a17 + 2 a19 + 2 a18 == 0 , a21 + 2 a20 + 2 a18 == 0, a22 + 2 a23 + 2 a18 == 0, a18 + 2 a24 + 2 a22 == 0, a25 + 2 a24 + 2 a21 == 0, a7 + 2 a26 + 2 a25 == 0, a5 + 2 a27 + 2 a25
      == 0, a25 + 2 a20 + 2 a19 == 0, a28 + 2 a29 + 2 a21 == 0, a21 + 2 a26 + 2 a28 == 0, a22 + 2 a27 + 2 a28 == 0, a28 + 2 a23 + 2 a19 == 0, a19 + 2 a29 + 2 a22 == 0, a7 + 2 a31
      + 2 a30 == 0, a32 + 2 a31 + 2 a33 == 0, a35 + 2 a34 + 2 a32 == 0, a39 + 2 a36 + 2 a32 == 0, a32 + 2 a37 + 2 a39 == 0, a33 + 2 a37 + 2 a35 == 0, a41 + 2 a40 + 2 a33 == 0,
     a33 + 2 a34 + 2 a41 == 0, a30 + 2 a38 + 2 a35 == 0, a42 + 2 a35 + 2 a39 == 0, a39 + 2 a43 + 2 a42 == 0, a44 + 2 a42 + 2 a43 == 0, a43 + 2 a45 + 2 a44 == 0, a11 + 2 a44 + 2
     a45 == 0, a45 + 2 a10 + 2 a11 == 0, a41 + 2 a38 + 2 a39 == 0, a30 + 2 a36 + 2 a41  == 0, a6 + 2 a40 + 2 a30 == 0}, {a1, a2, a3, a4, a5, a6, a7, a8, a9, a10, a11, a12,
     a13, a14, a15, a16, a17, a18, a19, a20, a21, a22, a23, a24, a25, a26, a27, a28, a29, a30, a31, a32, a33, a34, a35, a36, a37, a38, a39, a40, a41, a42, a43, a44, a45},
    Modulus → 5]
   ··· Solve : Equations may not give solutions for all "solve" variables.
Out[8]= {{a40 → a18, a38 → a18, a37 → a18, a36 → a18, a34 → a18, a31 → a18, a29 → a18, a27 → a18, a26 → a18, a24 → a18, a23 → a18, a20 → a18, a13 → a18, a8 → a18, a7 → a18, a6 → a18,
     a4 → a18, a2 → a18, a45 → a18, a44 → a18, a43 → a18, a42 → a18, a17 → a18, a16 → a18, a15 → a18, a14 → a18, a11 → a18, a10 → a18, a5 → a18, a41 → a18, a35 → a18, a33 →
     a18, a32 → a18, a30 → a18, a28 → a18, a25 → a18, a22 → a18, a21 → a18, a19 → a18, a12 → a18, a9 → a18, a3 → a18, a1 → a18, a39 → a18}}
```

Figure 9: Computation results of the Allen-Swenberg example in Wolfram Mathematica.

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