Detecting Causality by Using Alexander Quandles and Alexander-Conway Polynomial

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Abstract

The paper by Samantha Allen and Jacob H. Swenberg suggests that the Jones polynomial is likely able to detect causality in 2+1-dimension global hyperbolic spacetime; however, the Alexander-Conway polynomial cannot. The natural question that arises then is what extra information needs to be added to the Alexander-Conway polynomial so that it can also distinguish causality. In this paper, I used some of the Alexander Quandles for the connected sum of 2 Hopf links and the Allen-Swenberg link and obtained the result that it does not distinguish between the two links, so it cannot detect causality.

1 Introduction

In a globally hyperbolic spacetime X, which has (2+1) dimensions and is in the form of $\sum \times \mathbb{R}$, where $\sum$ is a Cauchy surface homeomorphic to $\mathbb{R}^2$, we can define N_X as the space that contains all light rays within X. These light rays can be represented using a solid torus. When a point $P \in X$ is considered, a light cone intersects $\sum \times \mathbb{R}$ in a circular curve, defining a knot in the solid torus (S_P of P). According to the Low Conjecture, as proved by Chernov and Nemirovski [VC20], two points P and Q are causally related if and only S_P and S_Q are linked within N_X. Therefore, link invariants that distinguish whether S_P and S_Q are linked within N_X can detect causality. Findings by Allen and Swenberg [Joy82] suggest that the Jones polynomial is likely able to detect causality, while the Alexander-Conway polynomial may be insufficient. They identified a link that relates to possibly causally connected events, which the Alexander-Conway polynomial was unable to distinguish. In this paper, I check whether the Alexander Quandle can distinguish the two examples of Allen-Swenberg.

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2 Quandles and Cocycles

2.1 Quandle [Ame07] or [Cro04]

A quandle is a set X with an operation $\triangleright$ satisfying the properties:

1. $x \triangleright x = x$ for all $x \in X$;
2. For all $x, y \in X$, there exists a z such that $x = z \triangleright y$
3. $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$ for all x, y, $z \in X$, which is called self-distributivity.

It is also implied that there is an inverse operation, $\triangleright^{-1}$, such that $(x \triangleright y) \triangleright^{-1} y = x$ for all x, $y \in X$. To define the knot quandle, we assign a letter for each arc. The relationship at each crossing is shown as below:

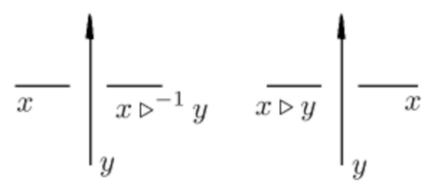

Figure 1: Quandles at crossings

The figure on the left shows that the arc that is labeled x crosses under the arc labeled y from left to right; therefore, the result is $x \triangleright^{-1} y$. The diagram on the right shows that the arc that is labeled x crosses under the arc labeled y from right to left; therefore, the result is $x \triangleright y$. To verify that the knot quandle is an invariant of knot, we check that the Reidemeister moves don't change the quandle colorings.

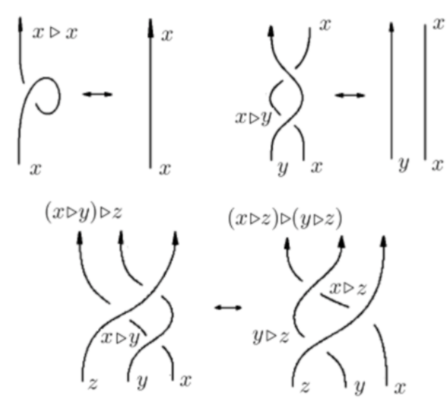

Figure 2: Quandles after Reidemester moves

If we have a knot diagram with labels on its arcs based on a quandle, these labels have a specific rule for the crossings. Before and after a certain move, there must be a bijection of the labelings. By comparing the number of labelings, we can determine whether the diagrams represent the same knot or different ones. If the numbers are equal, there's no conclusion; if the numbers are different, the diagrams correspond to different knots.

2.1.1 Fundamental Quandle

A fundamental quandle is a set of expressions used to represent a link through relationships at crossings. This makes the fundamental quandle a highly effective tool to compute quandle colorings because it serves as distinctive invariants for certain links.

2.1.2 Alexander Quandle

The Alexander quandle is an example of a quandle. It is constructed through the set of $\mathbb{Z}_n$ of integers modulo n and a t value which is co-prime to n. The quandle is defined by:

$$x \triangleright y = t\text{x} + (1 - t)\, y$$

If we do affine the Alexander quandle over $\mathbb{Z}_p$ then the cocycle would be $(x-y)^{pr}$. [CN10]

2.2 Colorings

A coloring is an assignment of elements from a quandle X to the arcs of a knot diagram, with the property that undercrossing is compatible with the $\triangleright$ operation.

2.3 Cocycles

Let X be a quandle, and take $A = \mathbb{Z}_n$ for some n. We want to enhance the coloring invariants using the notion of cocycle (which is part of the theory of Cohomology). A cocycle ϕ of X with coefficients in A is a function
$\phi : X \times X \to$ A satisfying the condition (for all x, y , z):
$\phi(x, z) + \phi(x \triangleright z, y \triangleright z) = \phi(x, y\ ) + \phi(x \triangleright y, z)$.

2.4 Boltzmann Weights

Let ϕ be a 2-cocycle of X with coefficients in A. Fix a coloring of a diagram D by X. At each crossing consider the element of A given by

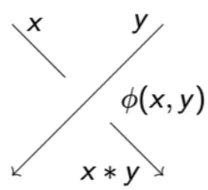

Figure 3: Cocycle relation at crossing

Define $B\phi(C) = \sum_{crossings} \pm\, \phi\ (x,\ y\ )$. This is called Boltzmann weight. The sign is defined $\pm$ 1 for positive and negative crossings, respectively. Then the cocycle invariant of the knot K (with diagram D) is given by the *formal sum* of Boltzmann weights:

ψ_ϕ(K) $= \sum_C\ B\phi$(C), where C runs over all colorings.

3 Causality Using Alexander Quandles

Let's calculate the Alexander quandle with $\mathbb{Z}_5$ and $t = 2$ for the connected sum of 2 Hopf links and the Allen Swenberg link.

$X \triangleright Y = 2X{-}1Y = 2X + 4Y$, mod 5.

If we do affine the Alexander quandle over $\mathbb{Z}_5$ then the cocycle would be $(x{-}y)^{5r}$. We first label all the crossings and arcs in the connected sum of 2 Hopf links diagram.

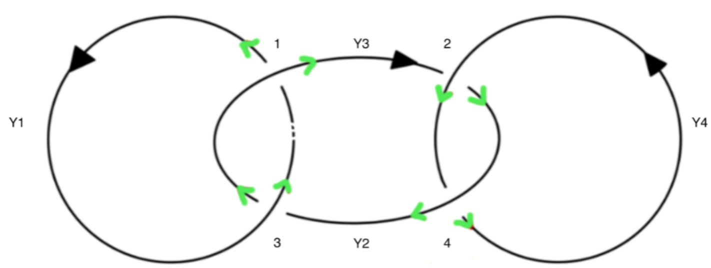

Figure 4: Connected sum of 2 Hopf links

Now, we can apply the Alexander quandle operation. The results are shown in the table below.

Crossings	Alexander Quandle
1	$Y_1 = Y_1 \triangleright Y_3 = 2Y_1 + 4Y_3$
2	$Y_2 = Y_3 \triangleright Y_4 = 2Y_3 + 4Y_4$
3	$Y_3 = Y_2 \triangleright Y_1 = 2Y_2 + 4Y_1$
4	$Y_4 = Y_4 \triangleright Y_2 = 2Y_4 + 4Y_2$

After solving the above system of equations, we obtain the following relation:
$Y_4 = Y_2$
$Y_1 = Y_2$
$Y_3 = Y_2$
Therefore, $Y_1 = Y_2 = Y_3 = Y_4$
Since all the colors are the same, it means that the number of colors is equal to the number of elements of $\mathbb{Z}_5$, which is 5.
We can apply the same process to the Allen Swenberg link. First, let's label all the crossings and arcs.

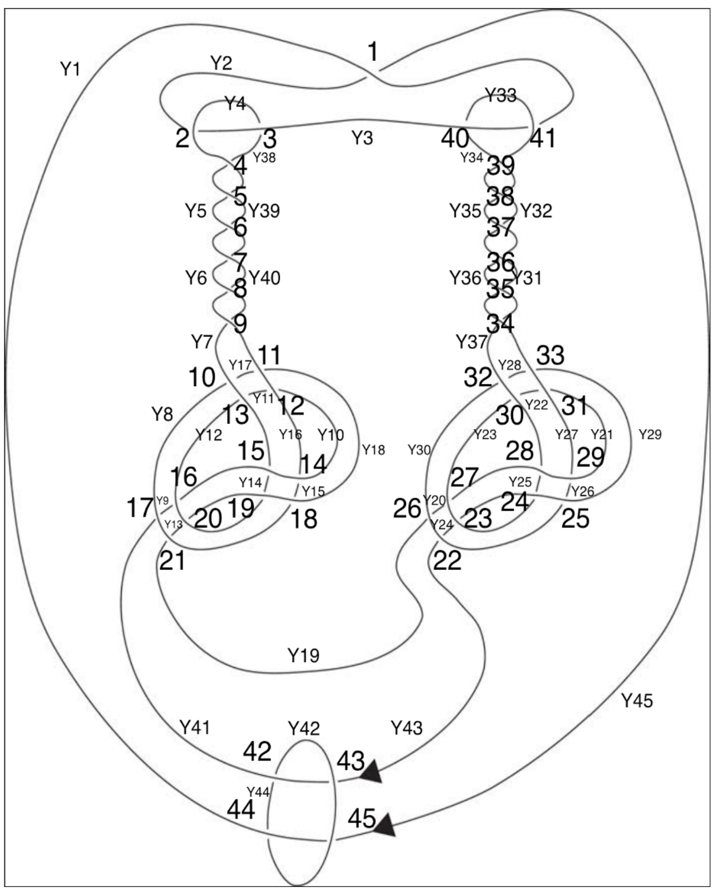

Figure 5: Labeled Allen-Swenberg link

Crossings	Alexander Quandle
1	$Y_{45} = Y_2 \triangleright Y_1 = 2Y_2 + 4Y_1$
2	$Y_2 = Y_3 \triangleright Y_4 = 2Y_3 + 4Y_4$
3	$Y_{38} = Y_4 \triangleright Y_3 = 2Y_4 + 4Y_3$
4	$Y_{39} = Y_{38} \triangleright Y_4 = 2Y_{38} + 4Y_4$
5	$Y_4 = Y_5 \triangleright Y_{39} = 2Y_5 + 4Y_{39}$
6	$Y_{40} = Y_{39} \triangleright Y_5 = 2Y_{39} + 4Y_5$
7	$Y_5 = Y_6 \triangleright Y_{40} = 2Y_6 + 4Y_{40}$
8	$Y_{16} = Y_{40} \triangleright Y_6 = 2Y_{40} + 4Y_6$
9	$Y_6 = Y_7 \triangleright Y_{16} = 2Y_7 + 4Y_{16}$
10	$Y_8 = Y_{17} \triangleright Y_7 = 2Y_{17} + 4Y_7$
11	$Y_{18} = Y_{17} \triangleright Y_{16} = 2Y_{17} + 4Y_{16}$
12	$Y_{10} = Y_{11} \triangleright Y_{16} = 2Y_{11} + 4Y_{16}$
13	$Y_{12} = Y_{11} \triangleright Y_7 = 2Y_{11} + 4Y_7$
14	$Y_{16} = Y_{15} \triangleright Y_{10} = 2Y_{15} + 4Y_{10}$
15	$Y_7 = Y_{14} \triangleright Y_{10} = 2Y_{14} + 4Y_{10}$
16	$Y_{10} = Y_9 \triangleright Y_{12} = 2Y_9 + 4Y_{12}$
17	$Y_{41} = Y_9 \triangleright Y_8 = 2Y_9 + 4Y_8$
18	$Y_8 = Y_{15} \triangleright Y_{18} = 2Y_{15} + 4Y_{18}$
19	$Y_{12} = Y_{14} \triangleright Y_{18} = 2Y_{14} + 4Y_{18}$
20	$Y_{18} = Y_{13} \triangleright Y_{12} = 2Y_{13} + 4Y_{12}$
21	$Y_{19} = Y_{13} \triangleright Y_8 = 2Y_{13} + 4Y_8$
22	$Y_{43} = Y_{24} \triangleright Y_{30} = 2Y_{24} + 4Y_{30}$
23	$Y_{29} = Y_{24} \triangleright Y_{23} = 2Y_{24} + 4Y_{23}$
24	$Y_{23} = Y_{25} \triangleright Y_{29} = 2Y_{25} + 4Y_{29}$
25	$Y_{30} = Y_{26} \triangleright Y_{29} = 2Y_{26} + 4Y_{29}$
26	$Y_{19} = Y_{20} \triangleright Y_{30} = 2Y_{20} + 4Y_{30}$
27	$Y_{21} = Y_{20} \triangleright Y23 = 2Y_{20} + 4Y_{23}$
28	$Y_{37} = Y_{25} \triangleright Y_{21} = 2Y_{25} + 4Y_{21}$
29	$Y_{27} = Y_{26} \triangleright Y_{21} = 2Y_{26} + 4Y_{21}$
30	$Y_{23} = Y_{22} \triangleright Y_{37} = 2Y_{22} + 4Y_{37}$
31	$Y_{21} = Y_{22} \triangleright Y_{27} = 2Y_{22} + 4Y_{27}$
32	$Y_{30} = Y_{28} \triangleright Y_{37} = 2Y_{28} + 4Y_{37}$
33	$Y_{29} = Y_{28} \triangleright Y_{27} = 2Y_{28} + 4Y_{27}$
34	$Y_{36} = Y_{37} \triangleright Y_{27} = 2Y_{37} + 4Y_{27}$
35	$Y_{27} = Y_{31} \triangleright Y_{36} = 2Y_{31} + 4Y_{36}$
36	$Y_{35} = Y_{36} \triangleright Y_{31} = 2Y_{36} + 4Y_{31}$
37	$Y_{31} = Y_{32} \triangleright Y_{35} = 2Y_{32} + 4Y_{35}$
38	$Y_{34} = Y_{35} \triangleright Y_{32} = 2Y_{35} + 4Y_{32}$
39	$Y_{32} = Y_{33} \triangleright Y_{34} = 2Y_{33} + 4Y_{34}$
40	$Y_{34} = Y_{33} \triangleright Y_3 = 2Y_{33} + 4Y_3$
41	$Y_1 = Y_3 \triangleright Y_{33} = 2Y_3 + 4Y_{33}$
42	$Y_{44} = Y_{42} \triangleright Y_{41} = 2Y_{42} + 4Y_{41}$
43	$Y_{41} = Y_{43} \triangleright Y_{42} = 2Y_{43} + 4Y_{42}$
44	$Y_{42} = Y_{44} \triangleright Y_1 = 2Y_{44} + 4Y_1$
45	$Y_1 = Y_{45} \triangleright Y_{42} = 2Y_{45} + 4Y_{42}$

To calculate the number of solutions, I inputted the system of equations into Wolfram Mathematica and obtained the following:

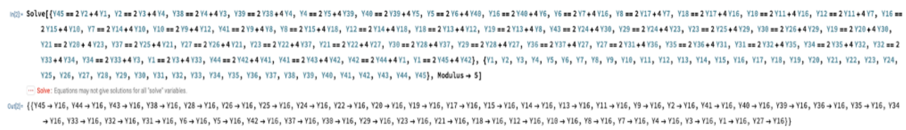

Figure 6: Solution provided by Wolfram Mathematica

The system shows that all the variables are equal to each other; therefore, the number of colors is equal to the number of elements of $\mathbb{Z}_5$, which is 5. Since the coloring invariant of the connected sum of 2 Hopf links is equal to the coloring invariant of the Allen Swenberg link, this invariant doesn't distinguish between the two links.

4 Using Other Values For t and n

I ran the same computations using different values for t and n and got the same number of solutions for both the connected sum of 2 Hopf links and the Allen Swenberg link.

$t = 3$, $n = 5 \to 5$ monochromatic solutions
$t = 4$, $n = 5 \to 5$ monochromatic solutions
$t = 2$, $n = 7 \to 7$ monochromatic solutions
$t = 3$, $n = 7 \to 7$ monochromatic solutions
$t = 4$, $n = 7 \to 7$ monochromatic solutions
$t = 5$, $n = 7 \to 7$ monochromatic solutions

In these cases, the number of solutions is equal to the n value of $\mathbb{Z}_n$, no matter the value of t. This is because we are finding trivial solutions for the colorings, and it means that there are exactly n number of colorings because $\mathbb{Z}_n$ has n elements.

5 Conclusion

The number of quandle coloring invariants for the connected sum of 2 Hopf links and Allen Swenberg link are the same for different values of n and t. The results show that the Alexander quandle paired with the Alexander-Conway Polynomial does not contain enough information to detect causality. Since we affine Alexander quandle over $\mathbb{Z}_5$, then the cocycle would $(x - y)^{5r}$. However, they would not help since according to my computations all the quandle colors are the same.

6 Acknowledgement

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