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A Proof of Seven Circles Theorem using Hyperbolic Geometry

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ABSTRACT

In this paper, we explore a non-conventional proof of the Seven Circles Theorem using several concepts from hyperbolic geometry. We attempt to represent the picture, claimed by the statement, in the Klein model—followed by the Poincar´e's hyperbolic disk model of hyperbolic space—in order to analyze the claim. We consider an ideal hexagon to have been formed by the points of intersection of each of the six inner circles and the ideal boundary. We then assume that there exists a non-ideal hyperbolic triangle that is formed as a result of intersections between the three main diagonals of the hexagon. We then go on to contradict this claim by proving that the area of the non-ideal triangle is zero.

Introduction

In 1974, Evelyn, Money-Coutts, and Tyrrell proposed a new theorem about the configuration of touching circles in euclidean geometry. It was referred to as the Seven Circles Theorem [1] which states that,

Suppose that we have a chain of five circles $S_1, S_2, \ldots, S_5$ each touching the preceding circle in the chain, and all touching a fixed circle C. There are then two circles S_6 which can be drawn touching S_5, S_1 and C, thus forming a closed loop of six touching circles, all touching C. Then one choice of S_6 is such that the three lines joining opposite pairs of points of contact on C are concurrent [Fig. 11].

Since the release of the [Proposition 1], there are several proofs of this result which have been published [1] [2] [3]. However, the capacity of each work is limited to Euclidean Geometry. For example, [3] utilizes the Ceva's Theorem in order to prove the statement.

In this paper, we explore a proof of the Seven Circles Theorem using the hyperbolic geometry [4]. There are several gaps in the argument and lemmas of the paper [4]. We have clarified some of the details of this proof with systematic set of lemmas. Our exposition starts from very basic definition of metric spaces and goes on to develop hyperbolic geometry and several models related to it. This paper is appropriate to wide range of mathematicians starting from high school students. All the advanced mathematical concepts employed in the proof are well-defined beforehand. We utilize several models of hyperbolic geometry in order to obtain a stronger proof of the result.

The structure of paper is defined as follows: Section (1) provides the definition of metric spaces; Section (2) provides preliminary information regarding hyperbolic geometry that is necessary to understand the proof; Section (3) utilizes hyperbolic disk models in order to validate several claims required to provide a concrete proof of the Seven Circles Theorem.

Metric Spaces

In the field of geometry, a metric is defined as a distance function; a space is a set with some structure; and a metric space, put together, is a set with a structure that is determined by a well-defined notion of distance. A quite familiar example of a metric space is the euclidean plane.

Definition 1. The euclidean plane is a set,

$$\mathbb{R}^2 = \{(x, y) |\ x, y \in \mathbb{R}\}$$

which consists of all ordered pairs (x, y) of real numbers x and y.

If γ is a curve in $\mathbb{R}^2$ which is parametrized by the differentiable vector-valued function

$$t \to \big(x(t), y(t)\big), a \leq t \leq b,$$

then, its euclidean length, $\ell_{euc}(\gamma)$, is the arc length given as,

$$\ell_{euc}(\gamma) = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt$$

where $\ell_{euc}(\gamma)$ is independent of the parametrization by a well-known consequence of the chain rule [5].

Definition 2. The euclidean distance, $d_{euc}(A, B)$, between two points A and B is the infimum of the lengths of all piecewise differentiable curves γ going from A to B [Refer to Fig 1]

$$d_{euc}(A, B) = \inf\{\ell_{euc}(\gamma) \mid \text{goes from } A \text{ to } B\}$$

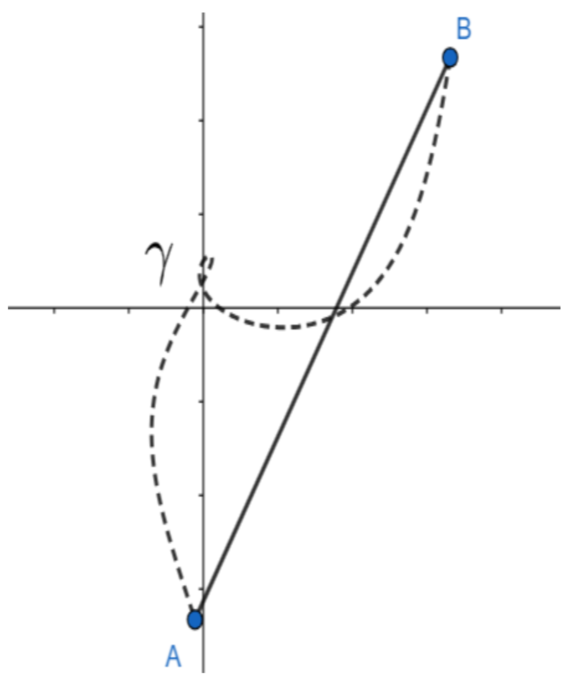

Figure 1. Notion of a path and distance in a two-dimensional euclidean plane

where, inf{ } implies that every piecewise differentiable curve γ going from A to B must have a length greater than or equal to $d_{euc}(A, B)$, and that there exist a set of curves whose length is arbitrarily close to $d_{euc}(A, B)$. In a two-dimensional euclidean space, a consequence of [Definition 2] is the well-known formula,

$$d_{euc}(x, y) = (\sum_{i=1}^{n} |x_i - y_i|^2)^{\frac{1}{2}}$$

where, x_i, y_i ($\in \mathbb{R}$) denote the coordinates of A & B in the plane.

The euclidean plane, $\mathbb{R}^2$, with its distance function, with its distance function $d_{euc}(A, B)$, is a well-known example of a metric space. However, the euclidean metric is not useful in every real-world scenario. For example, flights typically measure the distance between two locations along the great circle [Refer to Fig 2]—hence, the shortest distance between any two points is an arc along the sphere.

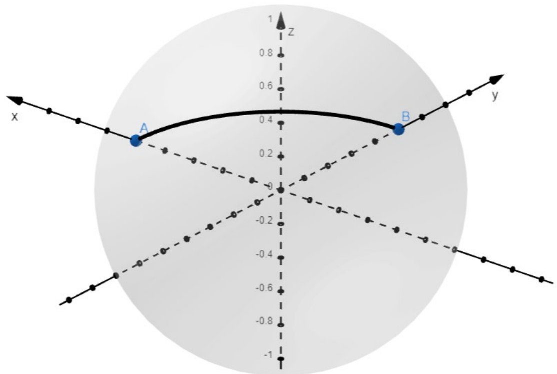

Figure 2. Shortest distance measured along the great circle

Metric space provides a setting for which the distance between several pairs of elements can be defined.

Definition 3. Metric Space is a pair (X, d) consisting of a set X together with a function $d : X \times X \rightarrow \mathbb{R}$ such that,

1. $d(x, y) \geq 0$ and $d(x, x) = 0$ for every $x, y \in X$
2. $d(x, y) = 0$ if and only if $x = y$
3. $d(y, x) = d(x, y)$ for every $x, y \in X$
4. $d(x, z) \leq d(x, y) + d(y, z)$ for every $x, y, z \in X$

In this scenario, (X, d) implies that d is a metric on the set X.

Hyperbolic Geometry

The hyperbolic plane, a model of hyperbolic geometry, is another example of a metric space, but it is much less familiar than the euclidean plane.

Discovery of Non-Euclidean Geometry

As discussed earlier, we are much familiar with the utilization of geometry (i.e. euclidean geometry) defined in a euclidean space. Euclidean geometry is widely studied because it approaches geometry in a systematic manner. In Euclid's book Elements [6], he proposed 23 definitions and 5 axioms based on which several theorems could be derived; as a result of this axiomatic approach, it led him to propose 465 propositions that described the entire euclidean geometry in a definitive order.

The most significant ideas of the book—that is—the five postulates proposed in the Book I include,

1. A straight line may be drawn from any point to any other point.
2. A finite straight line may be extended continuously in a straight line.
3. A circle may be described with any center and any radius.
4. All right angles are equal.
5. If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

Amongst the proposed postulates, the first four postulates were quite direct—but the fifth one, referred to as the parallel postulate, was complicated. [Fig 3] models the idea proposed in the fifth postulate.

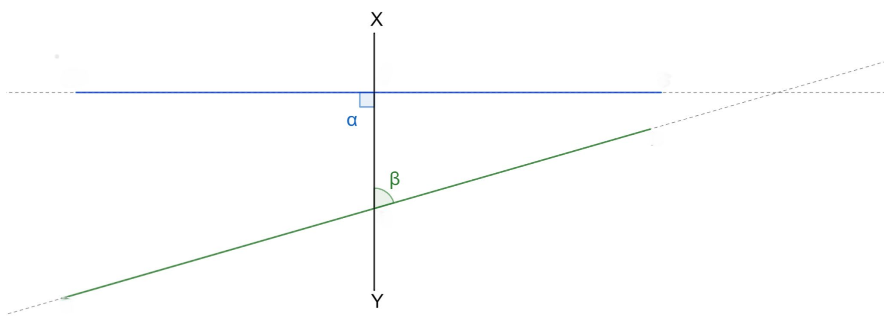

Figure 3. Euclid's Parallel Postulate

Based on the diagram provided, Euclid proposed that as the sum of two interior angles $\alpha + \beta$ approaches $180°$, the point of intersection between the two lines moves towards infinity; this postulate was claimed based on the definition of parallel lines quoted in, Book I, Definition 23,

"Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction."

In modern mathematics, the fifth postulate is typically replaced with the, simpler to comprehend, Playfair's Axiom which is an equivalent claim to Euclid's proposition in the presence of the other four postulates.

"Through a point outside a given infinite straight line there is one and only one infinite straight line parallel to the given line."

From a broad perspective, Euclid's attempt to describe geometry in its entirety was greatly admired. In fact, Elements, which was altogether published as a thirteen book series, went on to become the standard textbook of geometry that was used for over two thousand years.

However, even the earliest commentators of the book criticized the fifth postulate; the parallel postulate was long considered to be quite unnecessary. Over the course of years, many mathematicians asserted that the assumption made by Euclid is not sufficiently self-evident to be accepted without proof. Evidently, Euclid himself attempted to avoid its use till the first 28 propositions; nevertheless, he was forced to invoke it for Proposition 29 which states that,

"A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles."

For this theorem, he found it necessary to introduce the hypothesis in order to propose the claim.

It was long suspected that the fifth postulate could be derived from the other four postulates. In fact there were several attempts made by mathematicians, [Refer to [3]], to deduce the claim. However, it was not until the nineteenth century that the fifth postulate was finally shown to be independent of the other postulates of plane geometry. The proof of this independence coincidentally resulted in discovery of a completely new, consistent geometry, called Non-Euclidean Geometry.

Hyperbolic Geometry is an important part of the non-euclidean geometry which is utilized in the paper in order to provide a proof of the Seven Circles Theorem.

In the hyperbolic geometry—the parallel postulate suggests that,

"Through a point outside a given line there are infinitely many lines parallel to the given line."

This claim lays the motivation to study the hyperbolic plane as a useful metric space.

Hyperbolic Plane

Definition 4. Hyperbolic Plane, a model of hyperbolic geometry, is the metric space, $(\mathbb{H}^2, d_{hyp})$, consisting of the open half-plane,

$$\mathbb{H}^2 = \left\{(x,y) \in \mathbb{R}^2; y \geq 0\right\} = \{z \in \mathbb{C};\ \mathrm{Im(z)} > 0\},$$

where, the Imaginary part, $\mathrm{Im}(z)$, and Real part, $\mathrm{Re}(z)$, of a complex number $z = x + iy$ are the x and y coordinates respectively.

We define the hyperbolic length of a curve γ parametrized by the differentiable vector-valued function, mentioned in [Equation 1], as

$$\ell_{hyp}(\gamma) = \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)} dt.$$

Again, an application of the chain rule shows that this hyperbolic length is independent of the parametrization of γ [5].

Definition 5. The hyperbolic distance, d_{hyp}, between two points A and B is the infimum of the hyperbolic lengths of all piecewise differentiable curves γ going from A to B

$$d_{hyp}(P,Q) = \inf\{\ell_{hyp}; \gamma \text{ goes from } P \text{ to } Q\}$$

Isometries

An isometry of the hyperbolic plane is a mapping of the hyperbolic plane to itself that preserves the underlying distances. There are 3 special forms of isometries in the hyperbolic plane: (1) Homothety about points on $x-$axis, (2) Horizontal translation & Reflection about $y-$axis, (3) Inversion about point on $x-$axis.

Homothety about points on $x-$axis. A homothety h is a transformation defined by a center O and a real number k. It sends a point P to another point $h(P)$, multiplying the distance from O by k in the same direction. The number k is the scale factor. If the scale factor k is negative, then it takes the point P in the opposite direction. [Refer to Fig 4]

Remark 1. A homothety, about the origin—defined by the transformation $\varphi(x,y) = (\lambda x, \lambda y)$ for some $\lambda > 0$, is an isometry of $(\mathbb{H}^2, d_{hyp})$. A similar definition can be proposed for homothety about other points on the $x-$axis.

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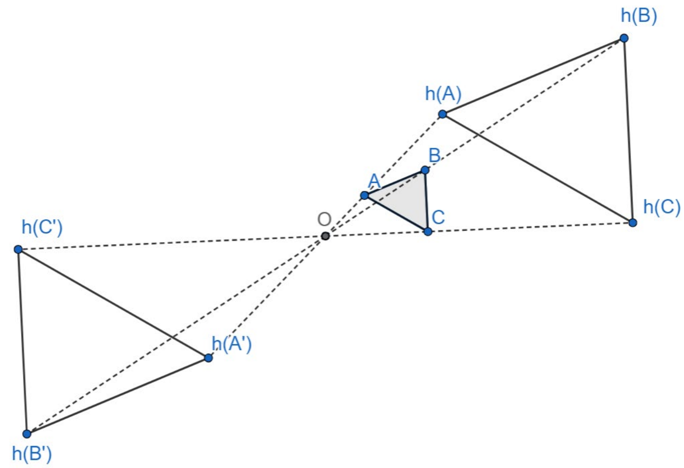

Figure 4. A homothety h with center O acting on ABC & $A'B'C'$

Horizontal translation & Reflection about y-axis. A horizontal translation defined by $\phi(x, y) = (x + x_0, y)$, for some $x_0 \in \mathbb{R}$, and the Reflection $\phi(x, y) = (-x, y)$ across the y −axis are both isometries of $(\mathbb{H}^2, d_{hyp})$.

Inversion across the Unit Circle. Consider every line, in a euclidean plane, as a circle with infinite radius. We add a point at infinity, P_∞, which h every ordinary line passes through (and no circle passes through). As a result, every choice of three distinct points determine a unique cline (that is a "circle" and "line"), while two ordinary points plus the P_∞ determine a line. With this said, let ω be a circle with center O and radius R. An inversion about ω [Refer to Fig 5] is a map which does the following [7]:

1. The center O of the circle is sent to P_∞
2. The point P_∞ is sent to O
3. For any other point A, we send A to the point A' lying on ray OA such that $OA \cdot OA^* = r^2$.

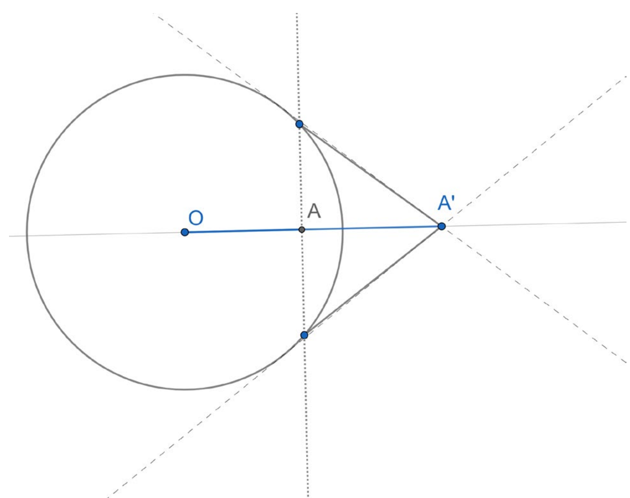

Figure 5. A' is the image of the point A when we take an inversion about ω

Remark 2. The inversion across the Unit Circle is an isometry of $(\mathbb{H}^2, d_{hyp})$ defined as,

$$\varphi(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right).$$

This map is typically represented in polar coordinates—in order to send the point $[r,\theta]$ to $\left[\frac{1}{r},\theta\right]$ in polar coordinates. [Refer to Fig 6]

A similar definition can be constructed for inversion about other points on the x −axis.

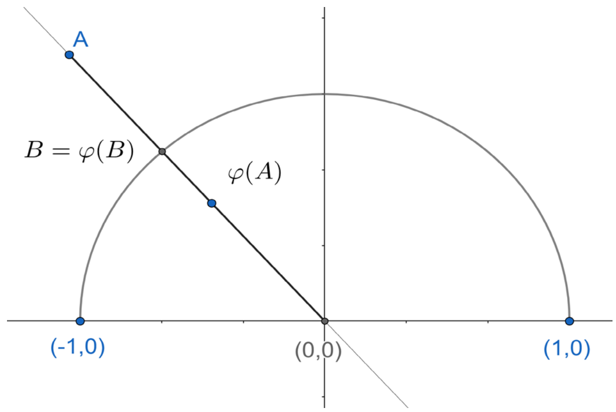

Figure 6. Inversion across the unit circle

In general, all isometries of $(\mathbb{H}^2, d_{hyp})$ are exactly the maps of the form

$$\varphi(z) = \frac{az+b}{cz+d}, \text{ where } a,b,c,d \in \mathbb{R} \text{ and } ad - bc = 1$$

or

$$\varphi(z) = \frac{c\bar{z}+d}{a\bar{z}+b}, \text{ where } a,b,c,d \in \mathbb{R} \text{ and } ad - bc = 1$$

Geodesics

Definition 6. A geodesic is a curve γ such that for every $P \in \gamma$ and for every $Q \in \gamma$ sufficiently close to P, the section of γ joining P to Q is the shortest curve joining P to Q (for the arc length considered)

Remark 3. Geodesic is a technical term for "shortest curve." For instance, geodesics in the euclidean plane $\left(\mathbb{R}^2, d_{euc}\right)$ are line segments.

There are 2 types of geodesics in the hyperbolic plane: (1) Vertical halfline, (2) Half-circle centered on the horizontal axis.

Vertical half-line. If $A_0 = (x_0, y_0)$, $A = (x_0, y_1) \in \mathbb{H}^2$ are located on the same vertical line, then the line segment $[A_0, A]$ has the shortest hyperbolic length among all piecewise differentiable curves going from A_0 to A [refer to Fig 7].

$$d_{hyp}(A_0, A) = \ell_{hyp}([A_0, A]) = \ln\left|\frac{y_1}{y_2}\right|$$

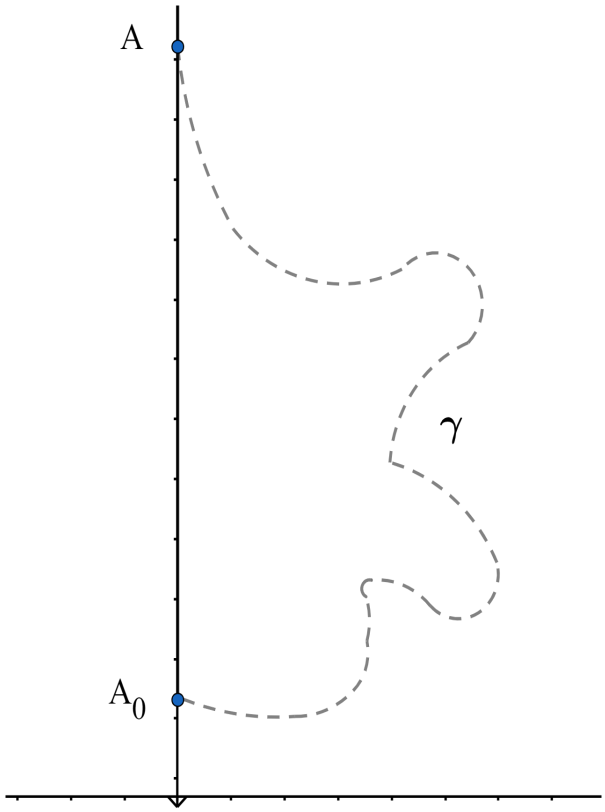

Figure 7. Geodesic 1: Vertical Line

Half-circle centered on the horizontal axis. Half-circle centered on the horizontal axis} For $A, B \in \mathbb{H}^2$ that do not lie on the same vertical line, have a semicircular arc connecting them as the unique shortest path [refer to Fig 8].

Remark 4. For any two points $A = (x_0, y_0),\ B = (x_1, y_1) \in \mathbb{H}^2$,

$$d_{hyp}(A, B) \geq \left| ln \frac{y_1}{y_0} \right|$$

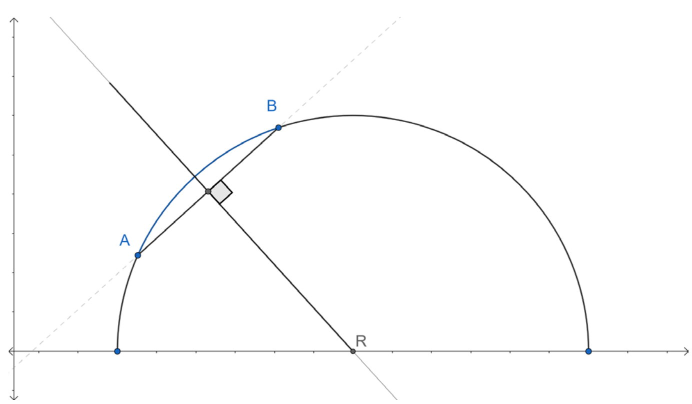

Figure 8. Geodesic 1: Circular Arcs

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Hyperbolic Disk

Definition 7. Hyperbolic Disk [4] is the set

$$B(z_0, r) = \{z \in \mathbb{H}^2 \mid \text{dist}(z, z_0) \leq r\}$$

where z_0 represents the hyperbolic center of the hyperbolic disk. Its boundary forms a hyperbolic circle [refer to Fig 9]

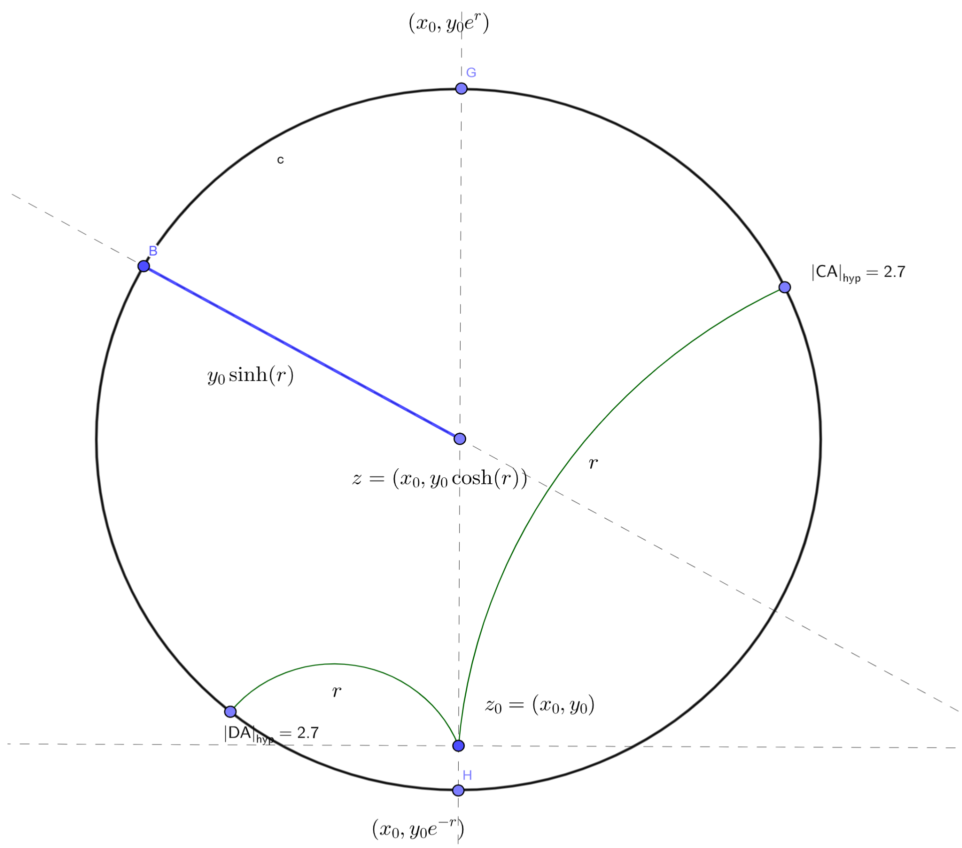

Figure 9. Hyperbolic Disk

Definition 8. Hyperbolic Circle is the locus of points equidistant from z_0.

Remark 5. Hyperbolic circle with center $z_0 = (x_0, y_0)$ and radius r is the euclidean circle with center $(x_0, y_0 \cosh(r))$ and radius $y_0 \sinh(r)$.

$\therefore$ Top and bottom points of the circle are $(x_0, y_0 \cosh(r) + y_0 \sinh(r)) = (x_0, y_0 e^r)$ and $(x_0, y_0 \cosh(r) - y_0 \sinh(r)) = (x_0, y_0 e^{-r})$, respectively.

Poincar é's Disk Model

Definition 9. Poincar é's Disk is a new model for the hyperbolic plane, namely, another metric space $(\mathbb{D}^2, d)$ which is isometric to $(\mathbb{H}^2, d_{hyp})$. It is represented as an open disk centered at the origin of a complex plane which is constructed by inverting the upper-half plane model about the point $(0, -1)$ with an inversion radius of $\sqrt{2}$.

Poincar é's Disk, denoted by $\mathbb{D}^2$, is a set whose points form an open disk

$$\mathbb{D}^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$$

Remark 6. boundary of the disk, $\partial\mathbb{D}^2 := x^2 + y^2 = 1$, is called as an ideal set.

Remark 7. The Hyperbolic Disk, which is a circle in the Hyperbolic plane, gets inverted to a circle in the Poincar é disk model. This circle is referred to as Hyperbolic Disk even in the Poincar é's model, that denotes the locus of equidistant points from a point.

Geodesics of the Poincar é's Disk

Lemma 1. Complete geodesics of the model are circular arcs orthogonal to the unit circle [Refer to Fig 10].

Proof. Angles in the Poincare model are preserved because inversion preserves angles. Complete geodesics in $(\mathbb{H}^2, d_{hyp})$ are orthogonal to the $x-$axis, which is considered as the boundary of the plane. Analogously, as a result of inversion which preserves the distances, complete geodesics of the disk model will be circular arcs orthogonal to the boundary, that is, unit circle.

Remark 8. [Lemma 1] also implies that the diameter of the unit circle is also a complete geodesic.

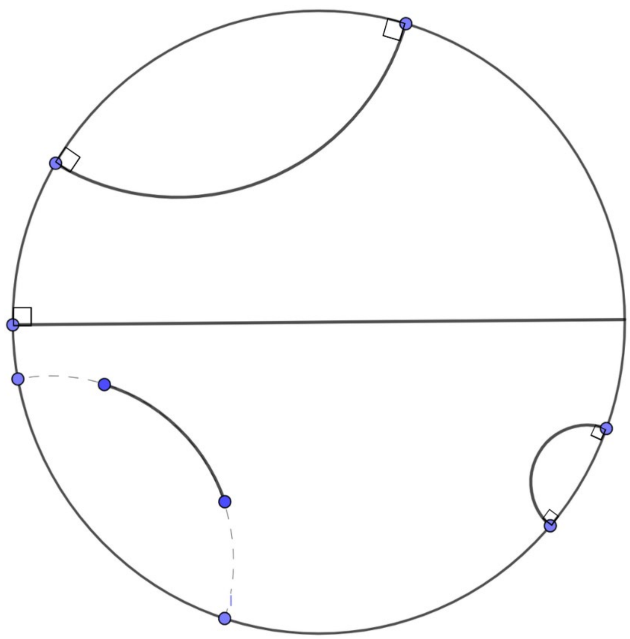

Figure 10. Geodesics of the Poincar é's Disk

Ideal & Semi-ideal Triangles

Lemma 2. An ideal triangle consists of three ideal points and the three hyperbolic lines connecting them [Refer to Fig 11].

Remark 9. All ideal triangles are congruent.

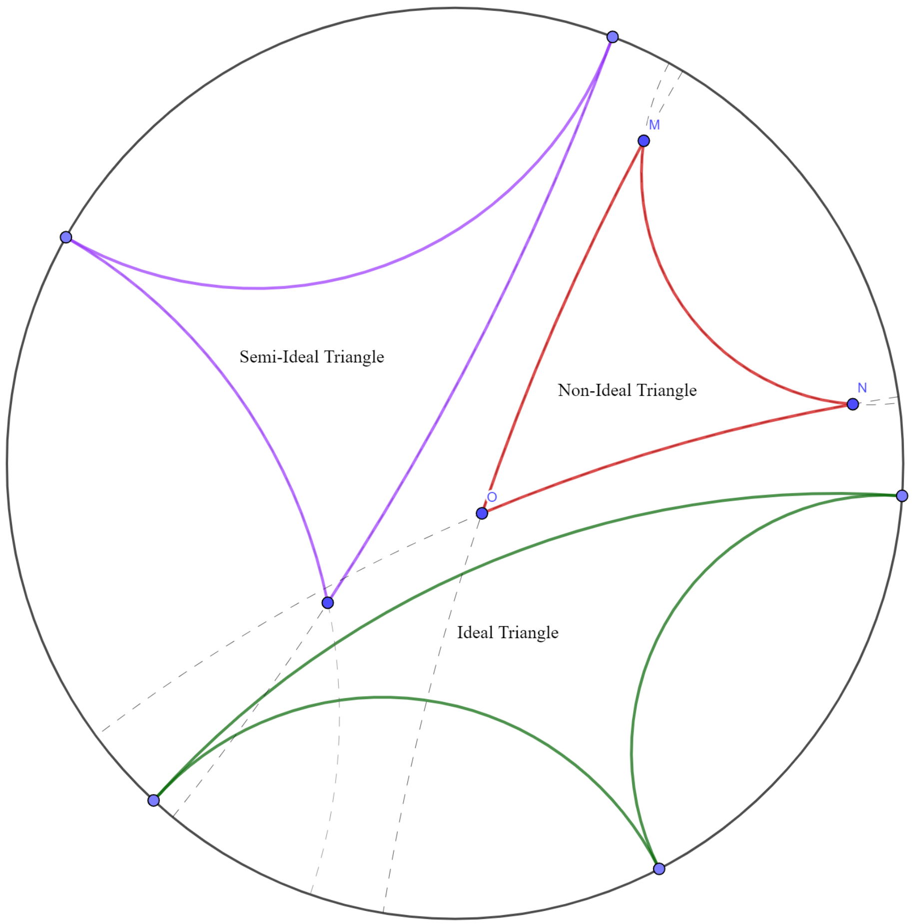

Figure 11. Ideal, Semi-Ideal & Non-Ideal Triangles

Remark 10. Similar to an ideal triangle, an ideal n −sided polygon in $\mathbb{D}^2$, is such that all its vertices are ideal points.

Horospheres

Definition 10. Consider a hyperbolic disk, Γ_1, with a center z_0 and radius r in the Hyperbolic plane, $\left(\mathbb{H}^2, d_{hyp}\right)$. When z_0 converges to the x −axis, the circumference of the circular disk becomes asymptotically tangent to the x −axis. Now, geodesics from z_0 to the circumference of the disk measure equal in lengths, though they are infinite in magnitude. The circumference of Γ_1 is called a horosphere [refer to Fig 12].

Remark 11. Notion of a horosphere can be extended to Poincare's Disk Model when z_0 asymptotically approaches the circumference of the unit-circle.

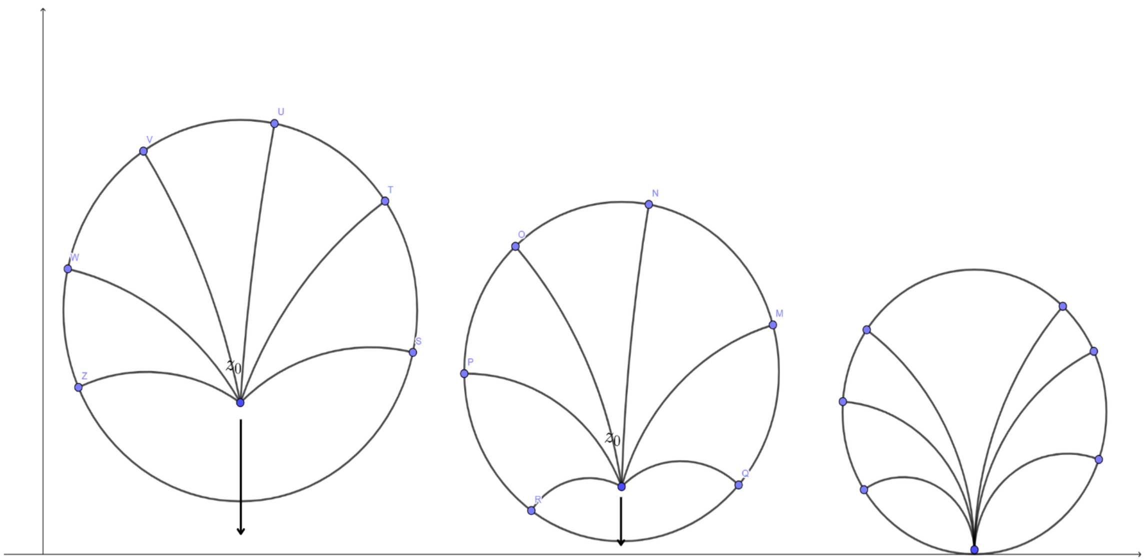

Figure 12. Notion of horospeheres in the upper-half plane model

Alternating Perimeter.

Definition 11. Hyperbolic polygon in $\mathbb{D}^2$ is a closed loop made by connecting together finitely many hyperbolic geodesic segments.

Definition 12. Given an even-sided hyperbolic polygon, P, in $\mathbb{D}^2$, let $S_2, \ldots, S_{2n}$ be the lengths of its consecutive sides. The Alternating Perimeter (AltPer) is defined to be the alternating sum of all lengths.

$$\text{AltPer} = (S_1 + S_3 + S_5 + \cdots + S_{2n-1}) - (S_2 + S_4 + S_6 + \cdots + S_{2n})$$

an alternate representation of this formula is,

$$\text{AltPer}(P_1) = \textstyle\sum_{i=1}^{n}(-1)^{i-1}S_i.$$

Consider a sequence P_n of a non-ideal hexagons that converge to form an ideal hexagon, that is, $P_n \xrightarrow{n\to\infty} P_\infty$. Put a horosphere at each vertex, P_n.

Lemma 3. AltPer of an ideal hexagon is equal, in magnitude, to that of a non-ideal hexagon.

Proof. Define $\widetilde{\ell_k}$ as the finite length of the edge between the borders of the disks. If we change the radius of the disk, we add and subtract the same amount from the alternating perimeter. Therefore, AltPer is independent of the radius of horospheres. When the radius of the disk is 0,

$$AltPer(P_1) \coloneqq \widetilde{\ell_1} - \widetilde{\ell_2} + \widetilde{\ell_3} - \widetilde{\ell_4} + \widetilde{\ell_5} - \widetilde{\ell_6} = A_1 - A_2 + A_3 - A_4 + A_5 - A_6$$

Klein-Beltrami Model

Klein-Beltrami model is another model of a hyperbolic space which is setwise equivalent to unit disk and its complete geodesics are the straight lines connecting the boundary points, that is, unit circle excluding the points on its circumference [Refer to Fig 13].

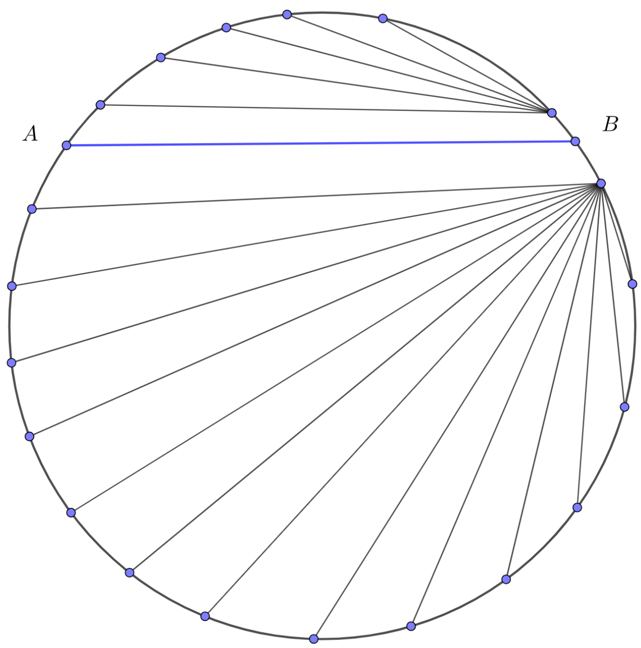

Figure 13. Klein-Beltrami model

Seven Circles Theorem.

Theorem. Let $A_0, A_1, A_2, A_3, A_4, A_5$ be six consecutive points around the circumference of a circle O in the euclidean plane. Suppose circles can be drawn internally tangent to circle O at these six points so that they are also externally tangent to each other in pairs (that is, the circle at A_i is tangent to the circle at A_{i-1} and the circle at A_{i+1}, where subscripts are reduced modulo 6). [See Fig 14] Then segments A_0, A_3, A_1, A_4 and A_2, A_5 concur.

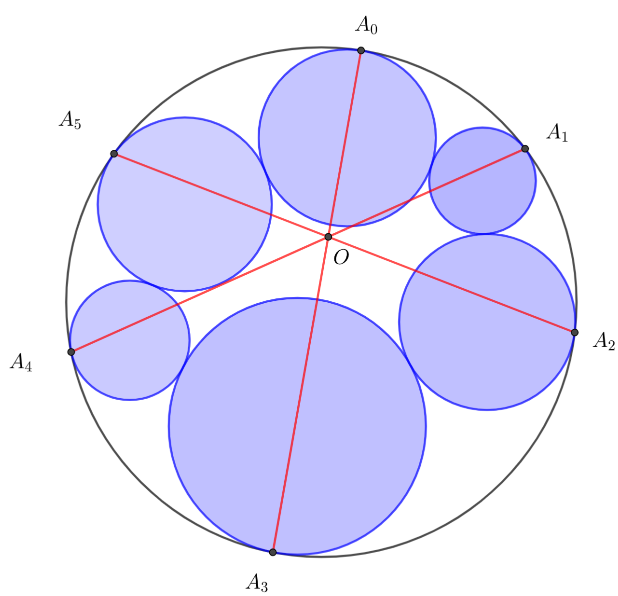

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Figure 14. Proposition of the Seven Circles Theorem in euclidean plane

Proof. We represent the hexagon $A_0, A_1, A_2, A_3, A_4, A_5$ in the Klein model and observe that the straight lines A_0, A_3, A_1, A_4 and , A_2, A_5, in the Klein model are the complete geodesics in it. Now, we will convert it into the Poincare's disk model and thus we have to essentially prove that these complete geodesics A_0, A_3, A_1, A_4 and , A_2, A_5, concur in the Poincar é disk model. After which we will be complete with the proof by converting the Poincar é 's disk model back to the Klein model [Refer to Fig 15].

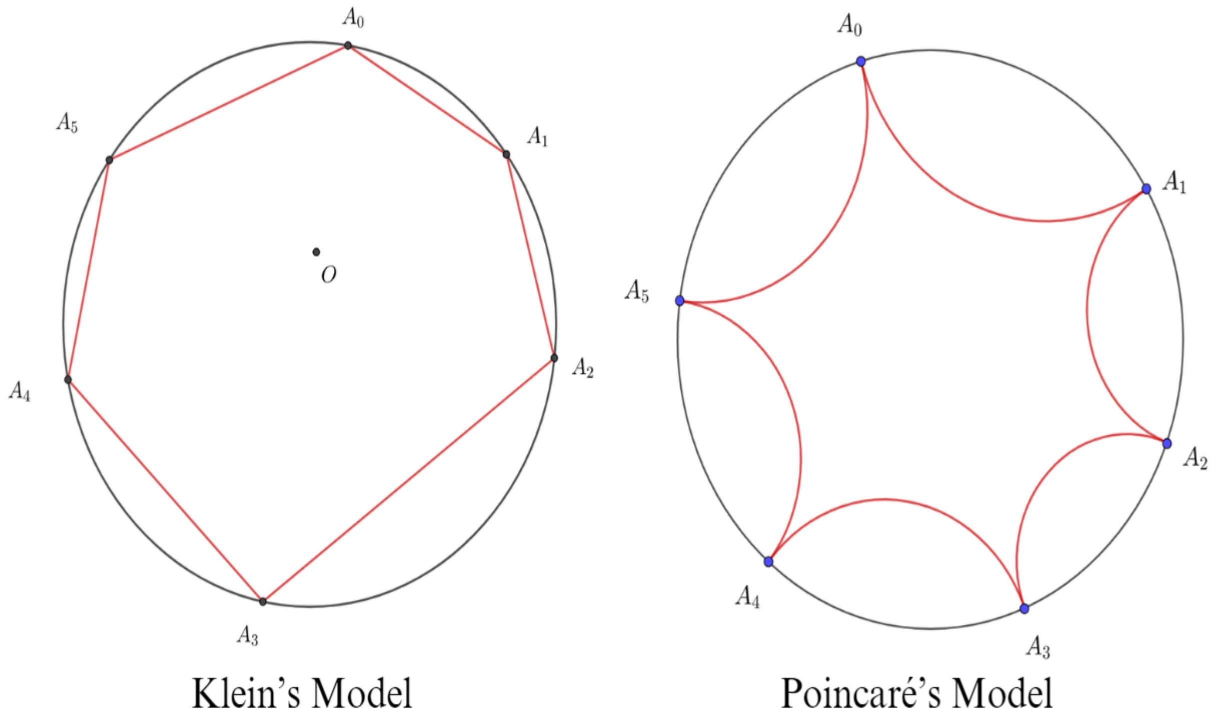

Figure 15. Transition from Klein's Model to Poincar é's Model

Lemma 4. Let Γ_1 & Γ_2 be two circles tangent to each other at R and to $\partial\mathbb{H}^2\}$ at P & Q respectively. Then, the geodesic from P to Q, denoted as $\widehat{PQ}$, must pass through R.

Proof. Let Γ_1 & Γ_2 have centers O_1 & O_2 respectively. Then O_1P & O_2Q must be perpendicular to the tangent to the set of circles and the Poincaré Disk at P and Q respectively.

Further, let L be the point of intersection between the lines tangent at P & Q to Γ_1 & Γ_2. Since, the geodesic is orthogonal to the Poincar'e Disk, the tangents at P and Q must pass through the center of the geodesic arc $\widehat{PQ}$. Hence, the center of the geodesic arc $\widehat{PQ}$ is L. Then, the line segments LP & LQ are the radii of the geodesic $\widehat{PQ}$.

The pairwise radical axes of three non-concentric circles are concurrent [Radical Axis Theorem], the radical axis between Γ_1 and Γ_2 must pass through L, which is the radical center. Thus, LR is the tangent to Γ_1 and Γ_2 implying $LP = LR = LQ$ and so point R lies on the geodesic $\widehat{PQ}$ [Refer to Fig 16].

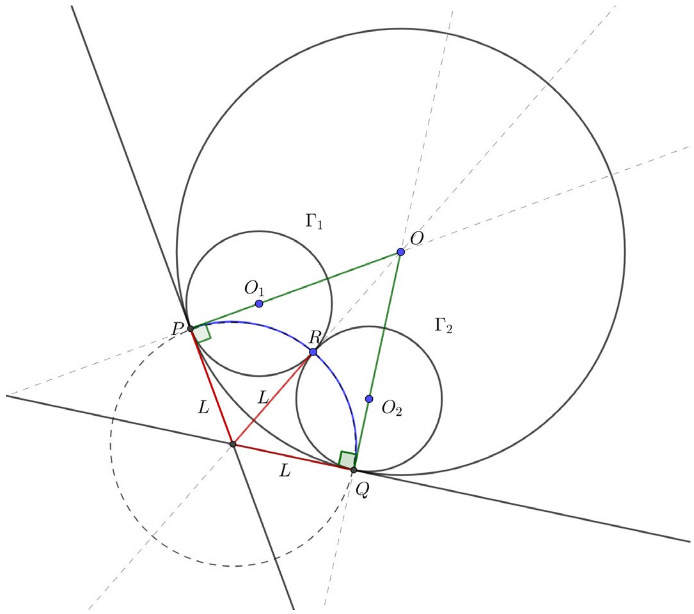

Figure 16. Representation of [Lemma 4]

As a result of [Lemma 4], we obtain the fact that there will be no portion of the geodesics in the ideal hexagon outside the horosphere.

Result 1. Alternating perimeter of the ideal hexagon, $A_0, A_1, A_2, A_3, A_4, A_5$, is zero [refer to Fig 17].

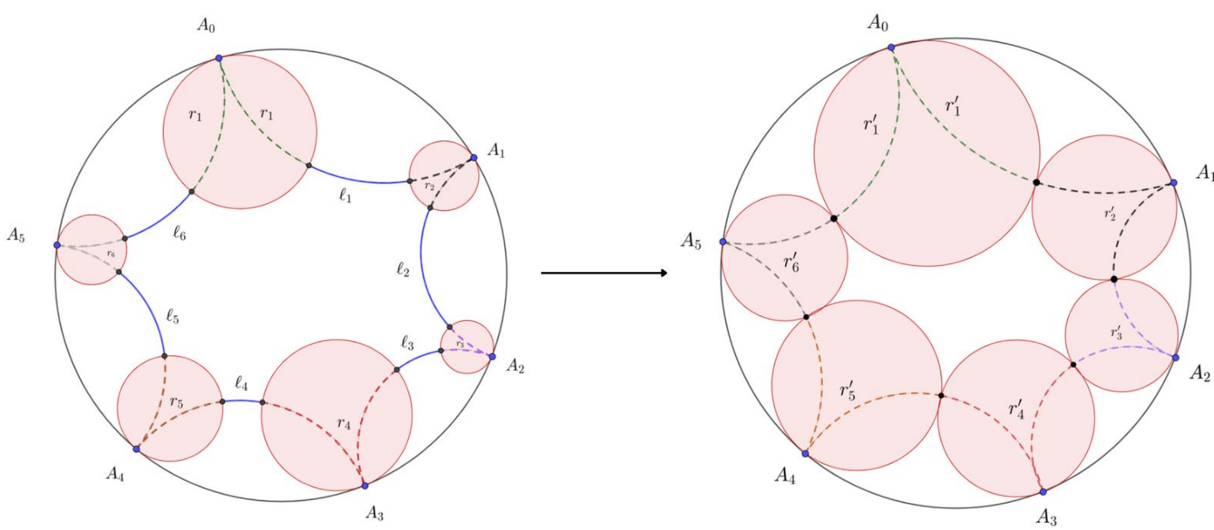

Figure 17. Change in magnitude of ℓ_n when the radius of hemispheres is increased

If we connect the opposite vertices of P, a triangle, they form the main diagonals, and which in turn intersect to form a triangle T_p, that is assumed to exist in the middle.

Lemma 5. For any ideal hexagon P,

$$|\text{AltPer}(P)| = \pm 2 \cdot \text{Per}(T_p)$$

Proof.

In P, let Y_1, Y_2, Y_3 be the semi-ideal triangles in $\mathbb{D}^2$. Three other triangles G_1, G_2, G_3, which overlap with T_P, are located opposite to corresponding Y_k triangles [Refer to 18].

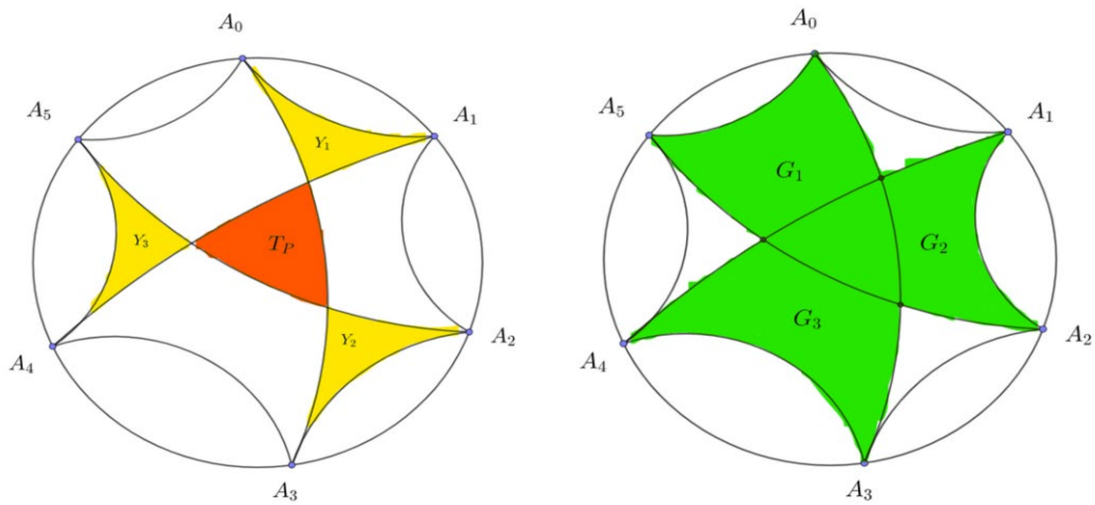

Figure 18. Ideal hexagon P and the small triangle T_P

Let L_1, L_2, L_3 denote the sides of any semi-ideal triangle V, where L_3 connects the two ideal vertices of V. Add disjoint horospheres at the vertices in order to obtain L_1', L_2', L_3' as a set of corresponding finite lengths of the triangle. Then, area of $P, A(V) = L_1' + L_2' - L_3'$.

Remark 12. The definition is quite similar to the notion of alternating perimeter but the number of sides is odd. The area does not depend on the radius of horospheres since the same length is both added and subtracted.

G_k and Y_k each share exactly one vertex for $k = 1,2,3$. In each case, there exists an isometry I_k such that $I_k(Y_k) = G_k$. $Y_k \mapsto G_k$ is a composition of three isometries, i, i', i'', which map the opposite triangles onto each other.

$$I_k = i \circ i' \circ i''$$

The order of transformation follows (1) mapping the intersection of the two triangles to the origin in order to obtain a straight line; (2) performing point reflection about the origin; (3) translating the intersection back to the original position.

Since $Y_k \mapsto G_k$ preserves area,

$$A(Y_k) = A(G_k).$$

We explicitly represent (13) for all three pairs of triangles.

$$A(Y_1) = A(G_1)$$
$$A(Y_2) = A(G_2)$$
$$A(Y_3) = A(G_3)$$

Let X, Y, Z be the points of intersections between the main diagonals.

Then,

$$[A(Y_1) = AX + FX - AF] + [A(Y_2) = BY + YC - BC] + [A(Y_3) = DZ + EZ - DE] = (AX + BY + DZ) + (FX + CY + EZ) + (-AF - BC - DE)$$

$$[A(G_1) = CX + XD - CD] + [A(G_2) = FY + EY - EF] + [A(G_3) = AZ + BZ - AB] = (AZ + BZ + DX) + (FY + CX + EY) + (-CD\_EF\_AB)$$

$\because$ (14) = (15), by rearrangement, we obtain

$$\mathrm{AltPer}(P) = (AX - AZ) + (BY - BZ) + (CY - CX) + (DZ - DX) + (FX - FY) + (EZ - EY)$$

Since,

$$(AX - AZ) = (DZ - DX) = -XZ$$
$$(BY - BZ) = (FX - FY) = -YZ$$
$$(CY - CX) = (EZ - EY) = -XY$$

the equation is represented as,

$$\mathrm{AltPer}(P) = -2(XZ - YZ - XY)$$

$\therefore$ We obtain,

Result 2. $|\mathrm{AltPer}(P)| = 2\mathrm{Per}(T_P)$

We utilize *Result 1 & 2* altogether, in order to obtain the proof of the Seven Circles Theorem.

Lemma 6. Let P be an ideal hexagon. Suppose that there are horodisks $H_1, \ldots, H_6$ placed at the ideal vertices of P in such a way that every two consecutive horodisks are tangent. Then the hyperbolic geodesics connecting opposite vertices of P concur at a point [Refer to Fig 19].

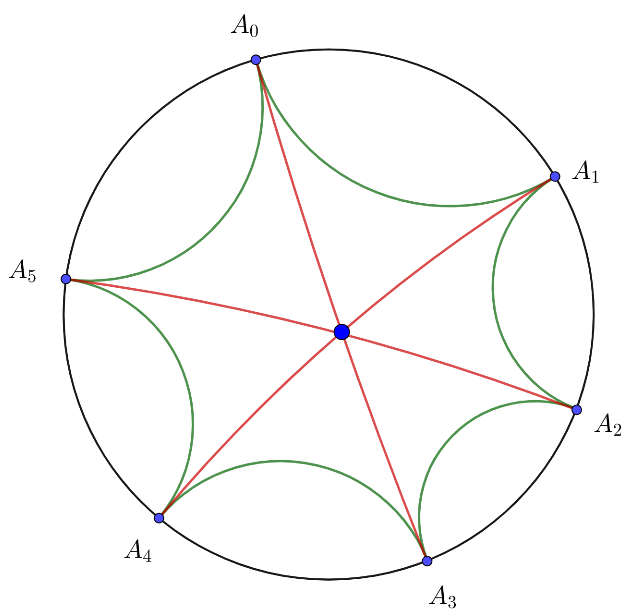

Figure 19. Opposite vertices of P concur at a point

Proof. Observe that the segments of hexagon in the circle cancel out. Therefore, the alternating perimeter is zero. As a result, the perimeter of the triangle in the middle is also 0.

$$A\text{ltPer}(P) = 0 = \pm 2 \cdot \text{Per}(T_P)$$

With this approach, we have finally proposed an exhaustive proof of the Seven Circles Theorem using hyperbolic geometry.

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