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Converting Subgroups of Generators of a Free Group into Free Generators Through Directed Graphs

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ABSTRACT

This research paper looks to address the following: Let H be a subgroup of $\mathbb{F}_2$. We are given a set of generators of subgroup H $\{h_1, h_2, \ldots, h_k\}$ which are not necessarily free. The goal is to find $\{h'_1, h'_2, \ldots, h'_s\}$, a set of free generators of H. We will first introduce the concept of groups, then we will introduce the concept of colored directed graphs, and we will finally connect the two to achieve this objective. We will address the problem through the use of Geometric Group Theory [1, 2, 3]. This will be done by converting {h1, h2, …, hk} into a colored directed graph, doing two different sequences of consecutive mappings known as folding and collapsing respectively, tracing back an edge path through these mappings, and converting these edgepaths back to words that will result in the free generators of H. In this article, we will explain all the relevant concepts, and work out a couple of examples.

Free Groups

In this section, we review the notions of abstract algebra that we will need in the rest of this paper. We explain the meaning: groups, subgroups, morphisms between groups, isomorphisms, generators of groups, free generators, free groups, and the rank of free groups.

Binary Operations

Given a set G, a binary operation on the set G is a function that takes as input pairs of elements of G (the order of the pair matters) and gives as output an element of G.

For example, addition is a binary operation on the set of integers. The input is a pair of numbers, and the output is the sum of those numbers. In this case, the order in of the numbers are in the input pair does not matter. For example, $3 + 5 = 5 + 3$. In cases like this one, where the order in of the numbers are in the input pair does not matter, we say that the operation is commutative.

An alphabet is a set whose elements are referred to as letters. For example, the lower-case letters *a* to *z,* form an alphabet. An example of a finite sequence of these letters is *abnear*.

Given two finite sequences of letters, we can create a third finite sequence of letters by placing the second sequence after the second one. This process is called concatenation. For example, the concatenation of the sequences *qwe* and *utf* is *qweutf.* Note that concatenation of finite sequences is a binary operation on the set of sequences. This operation is not commutative; the concatenation of *qwe* and *utf* but in reverse order would give *utfqwe* and not *qweuft*.

In the rest of this article, binary operations will be denoted by *, and the result of applying the binary operation to a pair of elements a and b will be denoted by $a * b$ or simply by ab.

Groups

A group is a set of elements G with a binary operation * that satisfy the following conditions:

1. $a * b$ belongs to G for all elements a and b in G.
2. $a * (b * c) = (a * b) * c$ for all elements a, b and c in G (associative property).
3. There exists an element e in G such that $a * e = a * e = a$ for all elements a in G (identity element).
4. For all elements a in G, there exists an element b in G such that $a * b = e$. This element b called the invers of a and is denoted by a^{-1} (i.e. $b = a^{-1}$).

For example, the set of integers, denoted by Z, with the addition is a group. The identity element is the number 0. The inverse of a number n is $-n$.

Words

Let a and b be two different elements (they can be anything). We usually refer to them as symbols or letters. From these two letters, we create two new letters that we denote by a^{-1} and b^{-1}. A word on the letters a and b is a finite sequence on the letters a, b , a^{-1} and b^{-1}. An example of a word on a and b is $bbaa^{-1}a^{-1}$.

A word is reducible if aa^{-1}, $a^{-1}a$, bb^{-1} or $b^{-1}b$ are part of the word. We say that a word is irreducible if the word is not reducible. An example of reducible word is $bbaa^{-1}aab^{-1}a$. An example of an irreducible word is $ab^{-1}a^{-1}bbb$.

We reduce a word by removing aa^{-1}, $a^{-1}a$, bb^{-1} or $b^{-1}b$ from the word. Given this definition, only reducible words can be reduced. For example, by removing aa^{-1} from the word $bbaa^{-1}aab^{-1}a$, we reduce it to $bbaab^{-1}a$.

We will apply this process of removing aa^{-1}, $a^{-1}a$, bb^{-1} or $b^{-1}b$ from the words many times, till we end up with an irreducible word. For example, the word $bba^{-1}a^{-1}aaab^{-1}ba$ reduces to $bbaa$.

Note that we can reduce a word in different ways but, if we continue reducing as long as it is possible, we always end up with the same irreducible word.

Note also that we could end up with an empty word. For example, $baa^{-1}b^{-1}$ reduces bb^{-1}, which reduces to a word with no letters. That is why we allow words to have no letters. We call this word the empty word.

We use the natural notation of $a^2 = aa$, $a^{-2} = a^{-1}a^{-1}$. We define other powers of a, as well as powers of b similarly. For example, $b^2aa^{-1}a^2b^{-1}a = bbaa^{-1}aab^{-1}a$.

Free Groups of Rank 2

Let F_2 be the set of irreducible words on the letters a and b. Reducing the concatenation of pairs of elements of F_2 all the way to an irreducible word is a binary operation of F_2 that makes F_2 a group, where the empty word is the identity element.

For example, $b^2a^2b^{-1}a * a^{-1}b^2 = b^2a^3$ (we first concatenated and then reduced).

To compute the inverse of a word, reverse the order of the letters of the word and change the sign of the exponents. For example, $(b^2a^2b^{-1}a)^{-1} = a^{-1}ba^{-2}b^{-2}$, which we can verify by noting that $b^2a^2b^{-1}a * a^{-1}ba^{-2}b^{-2} = e$, where e is the empty word.

Subgroups

A group H is said to be a subgroup of G iff all its elements can be found in G and if it forms a group under the same operation.

For example, in the group F_2, a subgroup could be G = $\{bba,\ ab^{-1}a^{-1}bbb,\ a^2b^{-1}\}$, as all elements in this group are irreducible words made up of a and b that also have a group under the operation of concatenation and subsequent reduction

Generators

A subset Q of G is said to generate G iff every element of G can be expressed through a combination (product) of all elements and inverses of Q. Such a subset Q are known as a **generator of G**.

For example, the subset Q = {a, b} generates F_2 as all irreducible words can be formed by the letters a, b , a^{-1} and b^{-1}.

Free Generators

The term free generator is defined as a generator that can't be multiplied with itself or other generators to obtain the identity element.

For example, the generating subset Q = $\{a,\ b\}$ is free for the group F_2 as there is no way to combine these generators in order to obtain the identity element. In contrast, the generating subset H = $\{a,\ a^2\}$ is not free for the group F_2 as $a^2 * a^{-1} * a^{-1}$ = e.

Note that inverses of the same element are not counted here, as any element can multiply with its inverse to get the identity element. The question is whether an element of subset Q can become the identity element by multiplying it with itself or the other elements and their inverses.

Morphisms

A homomorphism f: G→ H is a transformation of one group G into another group H preserves the relations of the first group.
In other words, a homomorphism follows the following rules for operation * in G and operation ♦ in group H:

f(a * b) = f(a) ♦ f(b).

An isomorphism is a bijective homomorphism, meaning that the relation between the two groups is one-to-one.

Colored Directed Graphs

In this section, we introduce colored directed graphs, which are a topological model that can be used to better visualize and work with free groups. We introduce directed graphs, edgepaths, tightening, the fundamental group, graph maps, folding, collapsing, n-roses, and isomorphisms between the Free group of rank two and the fundamental group.

Directed Graphs

Let $\Gamma(V,\ E)$ represent a directed graph: a set of edges E (each notated e_i) and vertices V (each notated v), with each edge having an assigned direction. Each edge can either be a loop (its terminal vertex is equal to its initial vertex) or a non-loop.

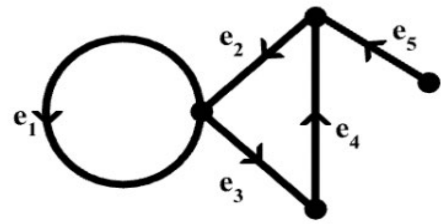

Figure 1. A directed graph Γ. The circles represent vertices while the edges are labeled.

Edgepaths

An edgepath is a string of edges $\alpha = e_1 e_2 ... e_k$ such that the terminal vertex of each edge corresponds to the initial vertex of the initial one of the next. In the case of an edge path, e^{-1} denotes traveling through aFin edge opposite to its assigned direction.

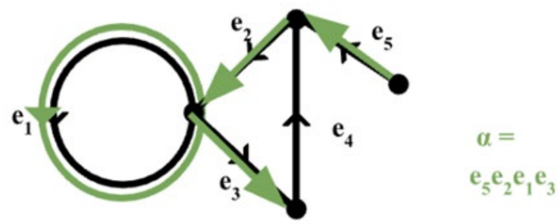

Figure 2. A directed graph Γ with an edgepath on top of it.

Closed And Tightened Edgepaths

If an edgepath is closed, this means that the terminal vertex of an edge path is the same as the initial vertex. It is said that the path is *based* on this vertex, *v*. In the example given in figure 2, the edgepath would be closed if the path started with e_4 instead of e_5.

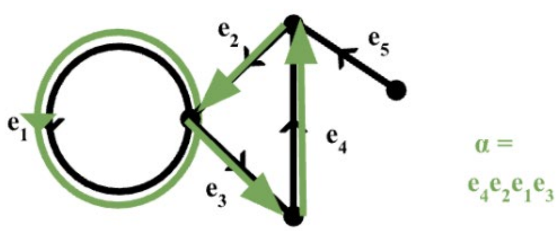

Figure 3. A closed, tight edgepath.

If an edge path is tight, it never traverses an element and immediately reverses course. Figure 2.3 would be tight as at no point in $\alpha = e_4 e_2 e_1 e_3$ is there an $e_i e_i^{-1}$ or $e_i^{-1} e_i$.

Tightening

A path is tightened through the following sequence:

Let $\alpha_0 = e_1 e_2 \ldots e_k$. If α_0 is not tight, then remove from the edge path one pair of $e_i e_{i-1}$ such that $e_i = e_{i-1}^{-1}$ to get α_1. Repeat this process until you get a tight path α_n.

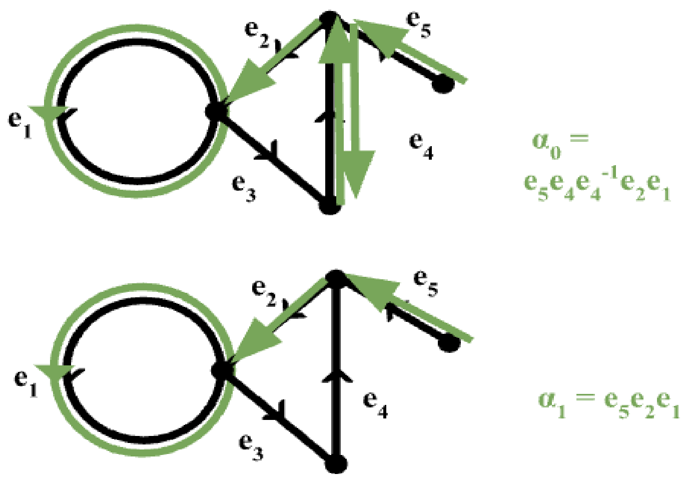

Figure 4. An edgepath is tightened by removing all pairs $e_i e_i^{-1}$ and $e_i^{-1} e_i$ from an edgepath.

The Fundamental Group

Under these conditions, the fundamental group $\pi(\Gamma, v)$ can be defined as the group of all closed, tight edge paths based at v for a directed graph Γ. The operation of the fundamental group is the concatenation of both elements and the tightening of the resulting edgepaths.

As an example, consider the directed graph Γ from Figure 2.1, but now with emphasis on vertex v.

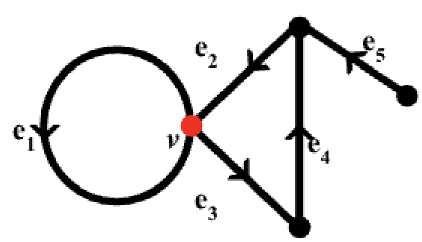

Figure 5. A directed graph Γ with a highlighted vertex v.

The fundamental group $\pi(\Gamma, v)$ for this directed graph Γ will then be all the paths that we can form in this directed graph that are closed, tight, and based at this vertex.

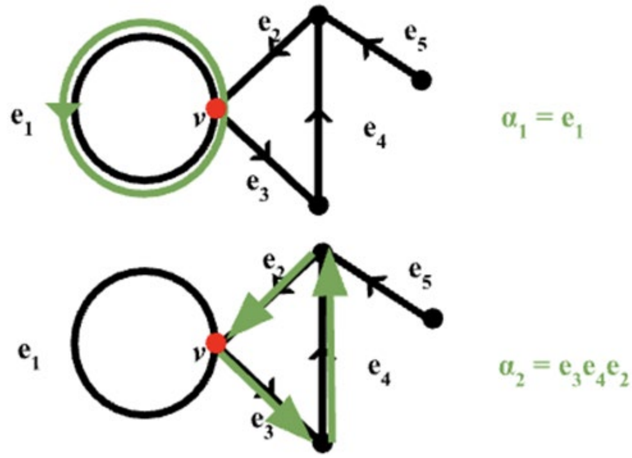

Figure 6. The tight, closed paths that can be taken based on vertex v.

The two paths depicted, plus the identity element of $\alpha = \emptyset$ in which there simply is no path, then conform the fundamental group for this directed graph. $\pi(\Gamma, v) = \{\emptyset, e_1, e_3e_4e_5\}$
In this case, the operation $e_1 * e_3e_4e_5 = e_1e_3e_4e_5$, with no tightening needed.

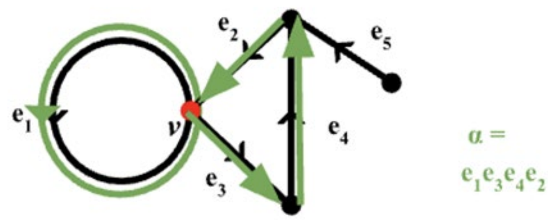

Figure 7. The edgepath resulting from the operation of both edgepaths in the fundamental group.

N-roses

An N-rose, denoted by $\mathbb{R}_n$, is a directed graph that has a single vertex v and n loops.

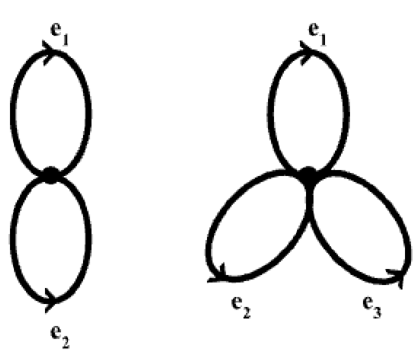

Figure 8. Directed graphs of $\mathbb{R}_2$ and $\mathbb{R}_3$ respectively

Free Groups, The Fundamental Group and Explanation of The Research Objective

At the beginning of this section, it was stated that directed graphs are topological models which help better visualize and work with free groups. Directed graphs and the fundamental group have now been covered in enough detail to introduce the isomorphism that connect the fundamental group to the free group of rank *n*.

Consider the following statement:

$$f: \pi(\mathbb{R}_n, v) \rightarrow \mathbb{F}_n \text{ is an isomorphism through the function } f(e_i) = a_i$$

Think of it like this: all the edgepaths that can be formed from an N-rose with *n* loops can be equated to all the words that can be formed through a combination of the *n* elements in the Free group by taking each edge of the n-rose and assigning it to an element of the Free group.

Figure 9. Directed graphs of $\mathbb{R}_2$

Consider figure 2.7. Now consider F_2 with generators *a* and *b*. If e_1 is mapped to *a* and e_2 is mapped to b, isn't the fundamental group's operation of the concatenation of both elements and the tightening of the resulting edgepaths equivalent to the Free groups operation of the concatenation of both elements and the reduction of the resulting word?

With this isomorphism in mind, consider the following task:

Let H be a subgroup of $\mathbb{F}_2$. We are given a set of generators of subgroup H $\{h_1, h_2, \ldots, h_k\}$ which are not necessarily free. The goal is to find $\{h'_1, h'_2, \ldots, h'_k\}$, a set of free generators of H.

The rest of this section will go through concepts which are necessary for explaining the task.

Graph Maps

Let $\Gamma(V, E)$ and $\Gamma'(V', E')$ be two different directed graphs. A graph map is a function $f: \Gamma(V, E) \rightarrow \Gamma'(V', E')$ that maps the vertices of edges of $\Gamma(V, E)$ to $\Gamma'(V', E')$.

It satisfies f(initial vertex of e_i) = initial vertex of e_i and f(terminal vertex of e_i) = terminal vertex of e_i. It is important to note non-loops can be mapped to loops but not the other way around.

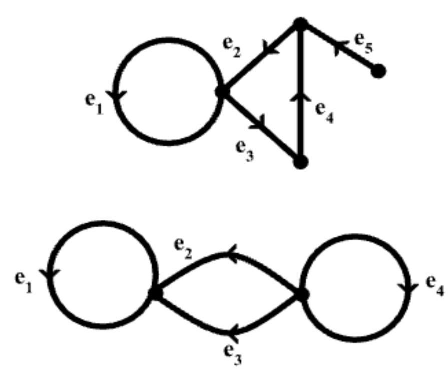

Figure 10. Graphs Γ(V, E) and Γ'(V', E').

Consider the two graphs in figure 2.6. A graph map f: $\Gamma(V, E)\rightarrow \Gamma'(V', E')$ could take the following form:
$f(v_{initial}$ of $e_3) = v_{final}$ of e_3 AND $f(v_{final}$ of $e_3) = v_{initial}$ of e_3
$f(v_{initial}$ of $e_4) = v$ of e_4 AND $f(v_{final}$ of $e_4) = v$ of e_4
$f(v_{initial}$ of $e_5) = v$ of e_4 AND $f(v_{final}$ of $e_5) = v$ of e_4
$f(e_5)$ = identity
Fi
All other f(x) are equal to x.

The following graph map has turned e_4 from a non-loop to a loop, which has in turn merged its two vertices. This means e_2 and e_3 now share both vertices. Also, notice how e_5 has been mapped to the identity, which means it is now skipped and therefore can be eliminated.

The important thing to note from graph maps is that they can induce homomorphisms between fundamental groups.

Specifically, consider the two directed graphs $\Gamma(V, E)$ *and* $\Gamma'(V', E')$ with graph map f: $\Gamma\rightarrow \Gamma'$. Then consider the fundamental groups $\pi(\Gamma, v)$ and $\pi(\Gamma, v)$.

It then follows that there is a homomorphism $f^\star$: $\pi(\Gamma, v)\rightarrow \pi(\Gamma', f(v))$ where the function $f^\star$ takes an edgepath $\alpha_0 = e_1 e_2 \ldots e_k$ and tightens it to an equivalent tight closed path.

Consider the following edgepath in the graph map proposed in figure 6.

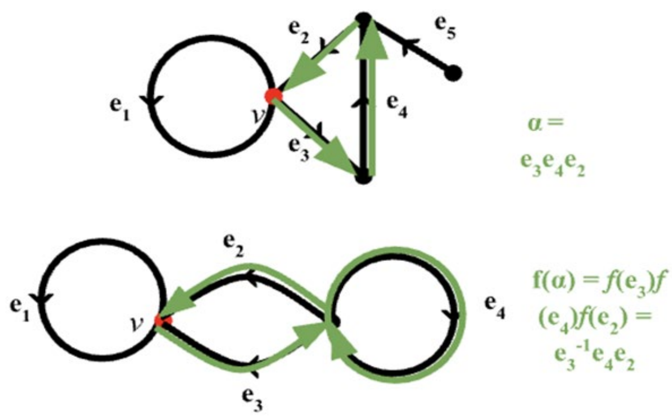

Figure 11. Graphs Γ and Γ′ with equivalent edgepaths.

Although these edgepaths are equivalent, the edgepath $f(\alpha)$ no longer belongs inside of $\pi(\Gamma', f(v))$, Fas it is not tightened. But if tightened as is shown in figure 2.8, the edgepath will now belong in $\pi(\Gamma', f(v))$. Even if the relation between the edgepath doesn't remain one-to-one, there is still a mapping.

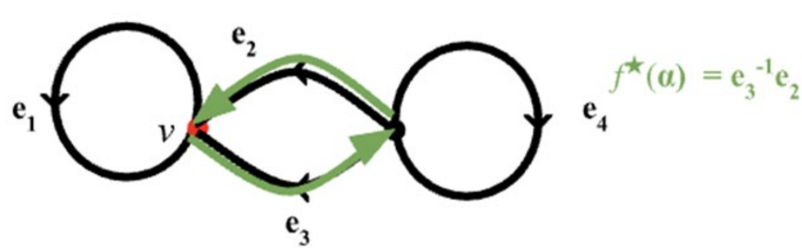

Figure 12. The edgepath $f^\star(\alpha)$.

A similar process can be followed for all edgepaths in all graph maps, which provides us with a homomorphism.

Folding

Folding involves taking two edges that share a vertex and merging them into a single edge. The following is the formal definition of folding.

Let Γ(V,E) be a directed graph, and let e_1 and e_2 be two different edges that share both orientation and at least one vertex. $\Gamma_{e1=e2}(V_{e1=e2}, E_{e1=e2})$ is considered to be the following:

I. The edges are now $E_{e1=e2}$ = E / {e_1 , e_2} ∪ {$e_{e1=e2}$}, where $e_{e1=e2}$ is a new edge that is a non-loop only if both e_1 and e_2 are non-loops.
II. The vertices are now $V_{e1=e2}$ = V / {vertices of e_1 , e_2} ∪ {vertices of $e_{e1=e2}$}.
III. For all e ∈ $E_{e1=e2}$ that share a vertex with $e_{e1=e2}$:
    A. If an initial or terminal vertex is shared with e_1 or e_2, the initial or terminal vertex is now shared with $e_{e1=e2}$.
    B. If they are not both initial or terminal vertices, then the terminal vertex stays as the terminal vertex and vice versa.

EXAMPLE 1. Consider this first directed graph:

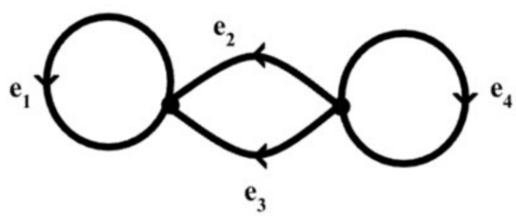

Figure 13. the directed graph Γ.

Suppose that we wish to fold edges e_2 and e_3. We would eliminate both edges and replace them with the single edge $e_{e2=e3}$ that connects to the vertexes that have replaced both of the existing ones. We are left with the following graph $\Gamma_{e2=e3}$:

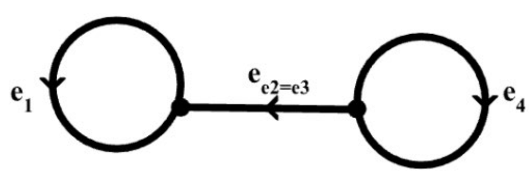

***Figure 14.** the directed graph* $\Gamma_{e2=e3}$.

EXAMPLE 2. Now consider the following directed graph:

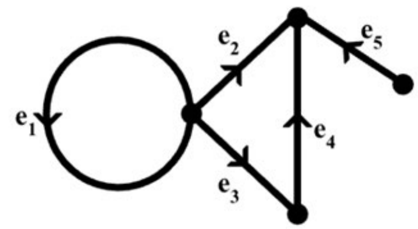

Figure 15. the directed graph Γ.

Suppose that we wish to fold edges e_2 and e_4. This would then join e_3's terminal vertex with its initial vertex as such:

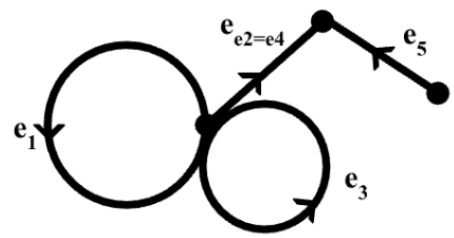

Figure 16. the directed graph $\Gamma_{e2=e4}$.

Suppose that we now wish to fold edges e_1 and e_3. This would leave us with the following graph:

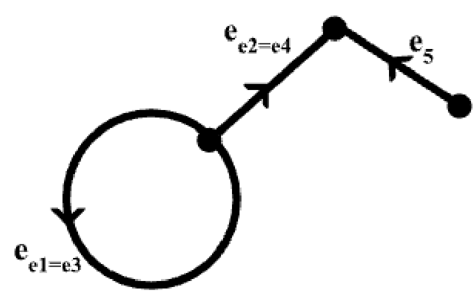

Figure 17. the directed graph of $\Gamma_{e2=e4 \text{ and } e1=e3}$ [Fi]

A directed graph Γ and the directed graph resulting from folding $\boldsymbol{\Gamma}$, $\Gamma_{e1=e2}$, can be related to one another through a graph map:

Let Γ (V, E) be a directed graph that has two edges e_1 and e_2 which can be folded. There is then a graph map $f_{e1=e2}$: $\Gamma \rightarrow \Gamma_{e1=e2}$ with the function $f_{e1=e2}(e_i) = e_i$ for $e_i \neq e_1$ or e_2 and $e_{e1=e2}$ for $e_i = e_1$ or e_2.
This then implies the homomorphism $f^\star$ that was introduced in the previous section:

$$f^\star_{e1=e2}: \pi(\Gamma, v) \rightarrow \pi(\Gamma_{e1=e2}, f_{e1=e2}(v))$$

In this case, the homomorphism $f^\star$ is also an epimorphism as it is surjective (every edgepath of $\pi(\Gamma_{e1=e2}$, $f_{e1=e2}(v))$ will have an equivalent version in $\pi(\Gamma, v)$). To understand why $f^\star$ is also an epimorphism, consider the following: Start with a tight path ***beta in*** $\Gamma_{e1=e2}$ in based at v' and then construct a tight path alpha such that $f^{\star}{}_{e1=e2}$ (alpha) = beta by replacing $e_{e1=e2}$ in the sequence beta with either e_1 or e_2 and $e_{e1=e2}{}^{-1}$ in the sequence beta with either $e_1{}^{-1}$ or $e_2{}^{-1}$. While the choice of whether e_1 or e_2 is not arbitrary and varies case by case, one of these two choices will always leave you with a path alpha. Therefore, while not injective, the epimorphism $f^{\star}$ is surjective.

Collapsing

In general, collapsing is taking a non-loop and destroying it. The following is the formal definition of collapsing:

Let Γ(V,E) be a directed graph and e be a non-loop. $\Gamma_{\downarrow e}(V_{\downarrow e}, E_{\downarrow e2})$ is considered to be the following:

I. The edges are now $E_{\downarrow e} = E / \{e\}$.
II. The vertices are now $V_{\downarrow e}$ = V / {endpoints of e} ∪ $\{v_{\downarrow e}\}$, where $v_{\downarrow e}$ is a new vertex.
III. For all $e_i \in E_{\downarrow e}$ that shared the initial or terminal vertex with e:
   A. The new initial or terminal vertex for e_i is now $v_{\downarrow e}$.

Consider the following example:

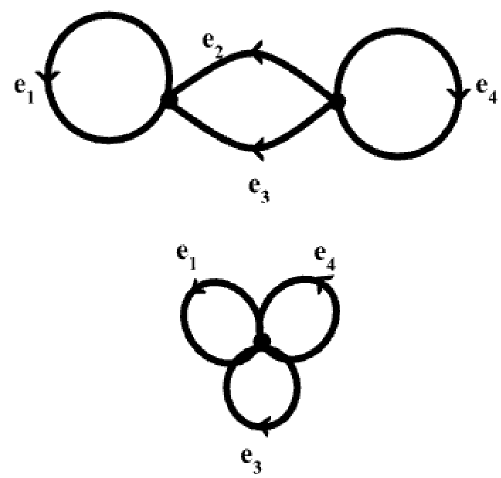

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Figure 18. The directed graphs of **Γ** and respectively $\Gamma_{\downarrow e}$.

Suppose we choose to collapse e_2. This would turn e_3 into a loop and leave the graph with a single vertex *v*, which is conveniently an N-rose.

A directed graph Γ and the directed graph resulting from collapsing ***Γ***, $\Gamma_{\downarrow e}$, can be related to one another through a graph map:

Let Γ (V, E) be a directed graph with an edge e that is not a loop. There is then a graph map $f\!\downarrow e$: $\Gamma \rightarrow \Gamma_{\downarrow e}$ with the function $f\!\downarrow e(v_i) = v_i$ for $v_i \neq$ endpoint of e and $v_i = v\!\downarrow e$ for v_i = endpoint of e.

This then leads to the following isomorphism:

$$f\!\downarrow e^{\star}: \pi(\Gamma, v) \rightarrow \pi(\Gamma, v_{\downarrow e})$$

Note that this relationship is one-to-one as any edge that is collapsed can be mapped to the identity in an edgepath, therefore implying that collapsing does not change the fundamental group of a graph.

Colored Directed Graphs

Colored directed graphs are directed graphs with assigned colors.

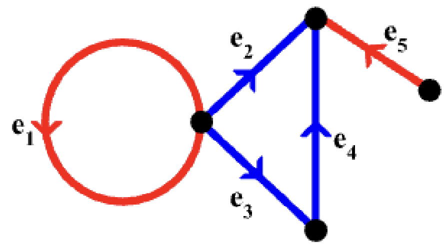


Figure 19. the colored directed graph Γ.

Task

In this section, the methodology for solving the task that is presented in the abstract and subsection 2.11. The task will be introduced more thoroughly along with the reasoning behind its pursuit, and the explanation of the methodology and why it works will be divided to in the following steps: Converting $\{h_1, h_2, ..., h_k\}$ into a colored directed graph; First sequence of consecutive mappings: Folding; Second sequence of consecutive mappings: Collapsing; Tracing edge-paths through the graph maps; Converting edgepaths back to free generators of H. For each of the steps, there will be an example that will be done through each part.

Relevance

It is relevant to convert a subset of generators into free generators as free generators simplify the algebraic description of subgroup H, allowing for a clearer understanding and manipulating of its structure. Many algorithms in computational algebraic structures, such as testing group membership, the word problem, and checking for isomorphisms, are easier to compute when the group in question is freely generated.

This can become critical in fields such as cryptography and coding theory, where such computations need to be efficiently. Without such efficiency, it would be much more difficult top to encrypt public-key cryptosystems, create hash functions, or construct certain type of error-correcting codes.

Converting $\{h_1, h_2, ..., h_k\}$ Into A Colored Directed Graph

Let H be a subgroup of $\mathbb{F}_2$. We are given a set of generators of subgroup H $\{h_1, h_2, ..., h_k\}$ which are not necessarily free. The goal is to find $\{h'_1, h'_2, ..., h'_k\}$, a set of free generators of H.

As it has been stated various times throughout the paper, directed graphs are a topological model used to better visualize and work with free groups. As such, our first step is to convert the subset of generators of $\mathbb{F}_2$ from a set of words to a colored directed graph.

Throughout this section of the paper, we will be working with the set of generators $\{a^2b^{-1}, a^3b, a^{-1}b^2, aba, a^3b^{-2}a^{-1}\}$

Note that as we will be using colored directed graphs, it will be useful to label the elements in terms of r (red) and b (blue). We therefore have the set of generators $\{r^2b^{-1}, r^3b, r^{-1}b^2, rbr, r^3b^{-2}r^{-1}\}$.

In order to turn a series of words composed of {r, b, r^{-1}, b^{-1}} into a colored directed graph, you will turn each generator into a closed path based at a single common vertex *v*. Look at the first letter of the word: if it is an inverse, make it clockwise, and if it is not, make it counterclockwise; if it is an r, paint the edge red, and if it is a b, paint the edge blue. Do this for every letter until you reach the end of the word at vertex *v*, then start the next one.

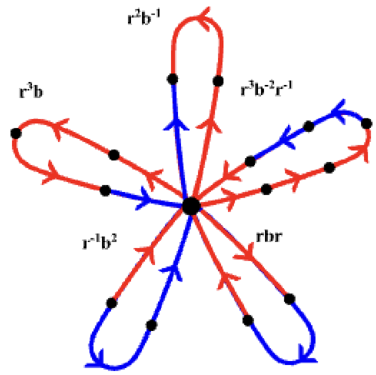


Figure 20. the directed graph resulting from the set of generators {r^2b^{-1}, r^3b, $r^{-1}b^2$, rbr, $r^3b^{-2}r^{-1}$}

Note that the directed graph given above does not have labeled edges. As this will only be necessary until the second sequence of consecutive mappings, we will avoid labeling the edges until then.

First Sequence of Consecutive Mappings: Folding

We can now go into the first sequence of consecutive mappings, which will be a series of folds. Envision the following:

A finite sequence of colored directed graphs Γ_0, Γ_1, ..., Γ_k will now be created. Each iteration of this sequence is a graph map $f_{e1=e2}$: $\Gamma \rightarrow \Gamma_{e1=e2}$ in which each edge in Γ is mapped to its corresponding edge in $\Gamma_{e1=e2}$, with the exception of e_1 and e_2 which will now be mapped to $e_{e1=e2}$. Continue this sequence until you reach a Γ_i in which there are no two edges that have the same color and initial or terminal vertex.

To understand the reason behind this initial sequence of mappings, look at the following diagram, in which the graph map $f_{e1=e2}$: $\Gamma \rightarrow \Gamma_{e1=e2}$ is contrasted with g:$\Gamma \rightarrow R_2$ and $g_{e1=e2}$:$\Gamma_{e1=e2}$

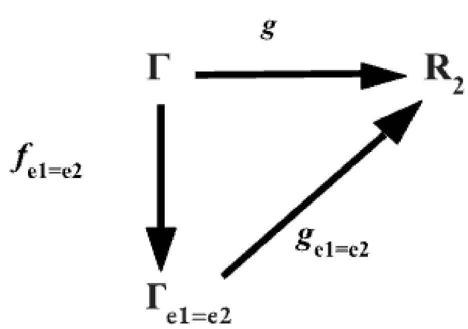


Figure 21. A factoring diagram for the graph map of $f_{e1=e2}$: $\Gamma \rightarrow \Gamma_{e1=e2}$

Given the graph map $\boldsymbol{f_{e1=e2}}$: $\Gamma \rightarrow \Gamma_{e1=e2}$, let g:$\Gamma \rightarrow R_2$ and $g_{e1=e2}$:$\Gamma_{e1=e2} \rightarrow R_2$ lead to the only colored graph map. Therefore $g_{e1=e2}$ composed of $f_{e1=e2}$ is equal to g. This therefore leads to:

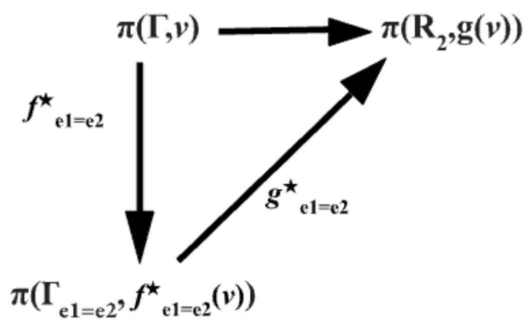
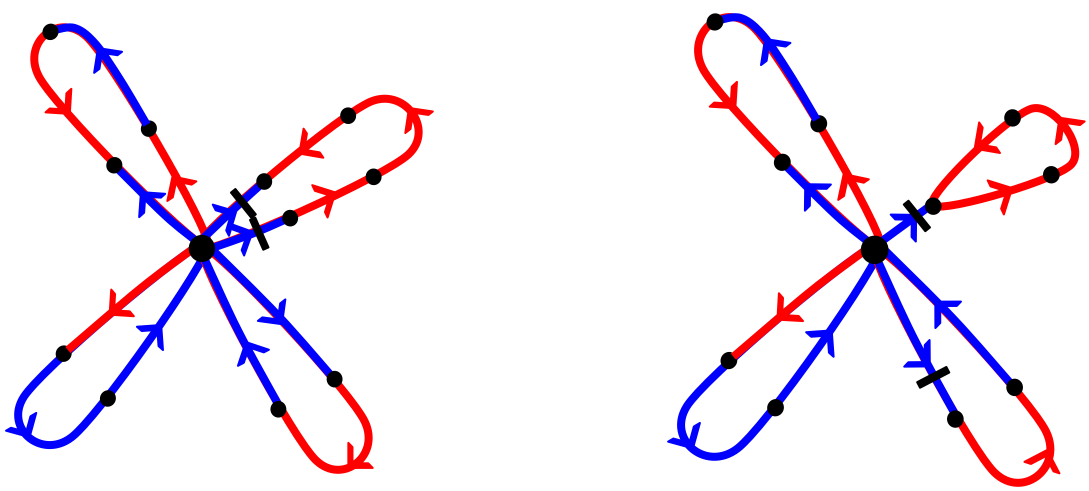
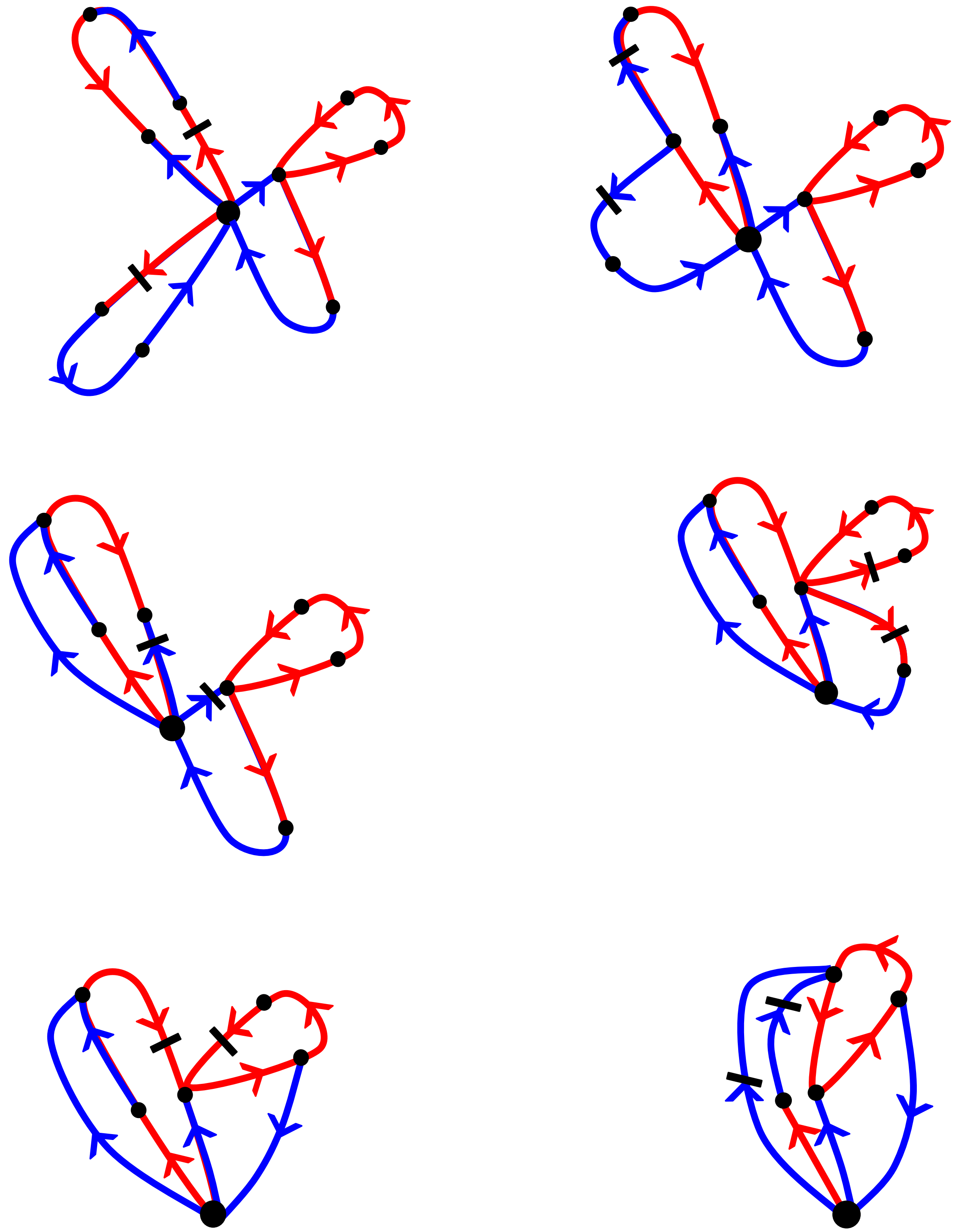


Figure 22. a factoring diagram for the graph map of $f^{\star}_{e1=e2}$: $\pi(\Gamma,v)\rightarrow\pi(\Gamma_{e1=e2}, f^{\star}_{e1=e2}(v))$

We can repeat mappings of $f^{\star}_{e1=e2}$ as many times as necessary in order to get to the desired change from the fundamental group of a random directed graph to a 2-rose.

We will now go over the folding sequence of figure 20. The two edges that are crossed out with a black mark are the ones that will be folded next.





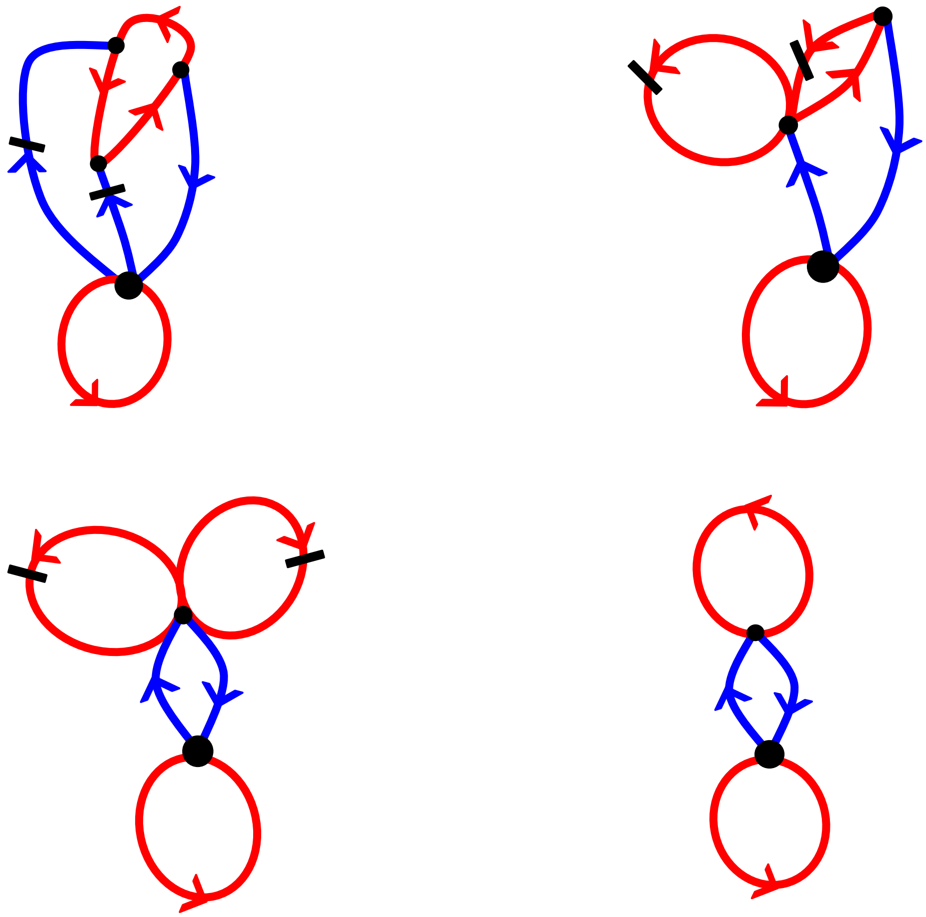


Figure 23. The first sequence of consecutive mappings, folding, done onto figure 20.

Second Sequence of Consecutive Mappings: Collapsing

Envision now a new finite sequence of colored directed graphs Γ'_0, Γ'_1, ..., Γ'_k, with each iteration of the sequence being a graph map $f^\star$: $(\Gamma', v') \rightarrow (\Gamma', v'_{\downarrow e})$ with $f^\star(\alpha) = \alpha_{\downarrow e}$. This sequence ends when there are no more edges to collapse.

We will now go over the collapsing sequence of figure 23. It is now pertinent to assign each edge a label.

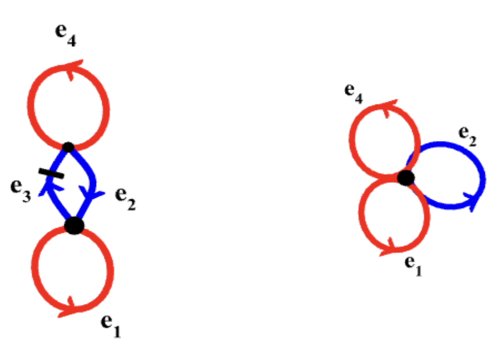


Figure 24. The second sequence of consecutive mappings, collapsing, done onto figure 23.

Tracing Edgepaths Through the Graph Maps

These loops are our free generators of $\pi(\Gamma'_k, v'_k)$. But in order to get the free generators of $\pi(\Gamma', v')$, we need to trace back the tight closed path based at v' in Γ'_0 for each e_i that map the edge as part of Γ'. See the following example.

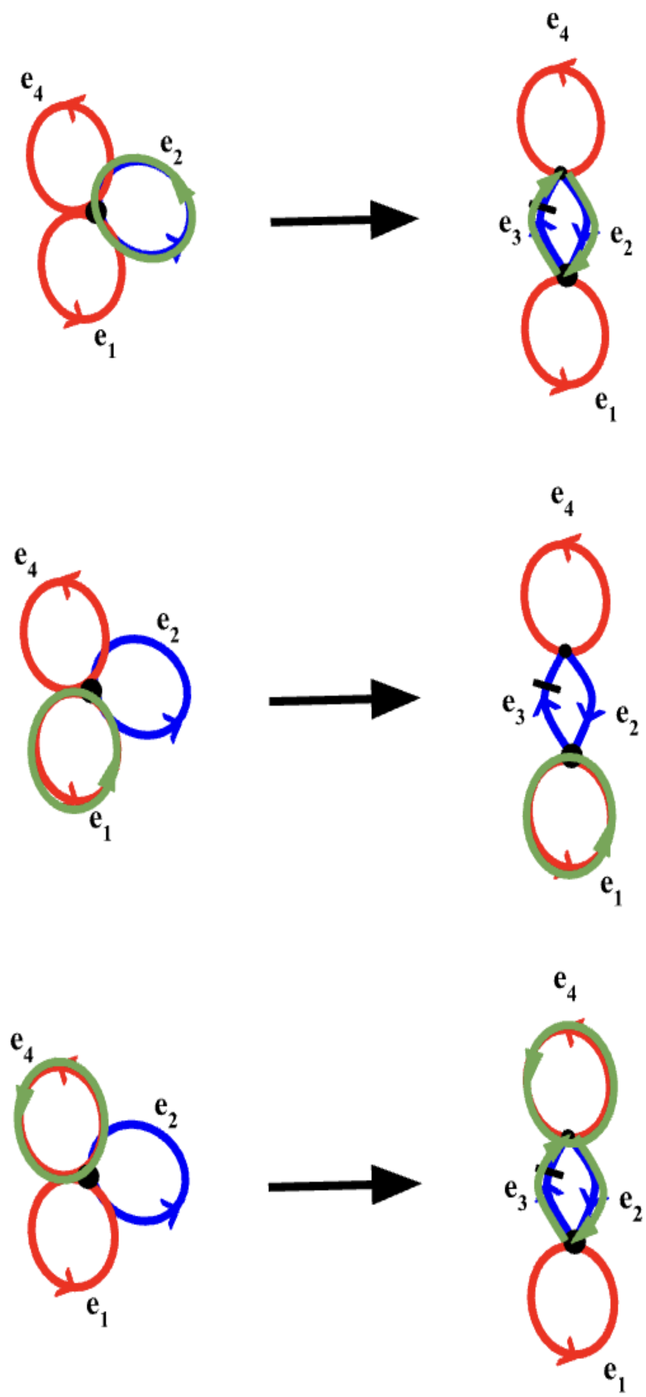


Figure 25. The tracing of edgepaths through the graph maps of figure 23.

Edgepath in Γ'_k	**e_1**	**e_2**	**e_4**
Edgepath in Γ'_0	**e_1**	**e_3e_2**	**$e_3e_4e_2$**

Figure 26. A summary of the traced edgpaths of figure 24.

Converting Edgepaths Back to Free Generators Of H

Now that you have the edgepaths, you can simply convert them back to convert them back to words in {r, b, r^{-1}, b^{-1}} in order to have our set of free generators. To do that, check whether each edge is either red (r), or blue (b) and counterclock wise (normal) or clockwise (inverse).

In this case, {e_1, e_3e_2, $e_3e_4e_2$} will therefore turn into {r, b^2, brb}.

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