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A Way of Proving Colinear Relationship Using Ratio Properties of Parallel Lines and Other Theorems

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ABSTRACT

Plane geometry has long been an essential part in Mathematics since the subject was born. Though it is no longer appealing to most researchers, plane geometry problems are now used in mathematics Olympics to test students' ability and countless minds are still exploring the beauty and possibilities created by two-dimension geometries. The author independently discovered a useful method of solving geometric problems and has successfully applied it on three quite different problems that had appeared in mathematics Olympics, which indicate that the method is not a coincidence or a flashy skill, but a sharp observation of a common configuration: parallel. Below, the method will be introduced more comprehensively, and some deeper understanding of it will also be revealed. First, the discovery of the method will be briefly explained in a solution. Then, two applications will be stated demonstrating broader uses of the method and rich properties related. Eventually, the pattern of intersecting lines and parallel lines will be further described with several relating theorems.

Introduction

First, an important conclusion is found by the writer which reveals good properties of parallel lines and associate different theorems and perspectives in plane geometry together. Given $\mathrm{AB}//\mathrm{CD}//\mathrm{EF}$, let $\mathrm{P}=\mathrm{AC}\cap\mathrm{BD}, \mathrm{Q}=\mathrm{AE}\cap\mathrm{BF}, \mathrm{R}=\mathrm{BE}\cap\mathrm{CF}$, then it can be inferred that P, Q, R are colinear. To prove this, we can view the problem from 3 different perspectives, each gives a unique explanation.

(1) Analysis of ratio relationships
This is the intrinsic reason that the conclusion is correct.
(2) Desargues theorem
This little conclusion is just a special case of Desargues theorem since three parallel lines can be seen to meet at point at infinity.
(3) Homothetic centers
In fact, this was the perspective the writer used when the writer discovered the method. It came from the observation that parallel segments are homothetic figures. Therefore, the conclusion is equivalent to the property of homothetic centers that the three homothetic centers of each two of three given figures are colinear if they are correspondent.

It must be accentuated that the method the writer found (which will be referred to as the method for short in the following paragraphs) is not simply the conclusion above, it is a way to use it and combine it with other theorems in problem solving that can apply for a wide range of situations. This conclusion is just a step in the method, and other theorems like the Monge theorem, the Pappus theorem can also have similar effects.

In problem 1, the method is discovered and gave a beautiful solution.

In problem 2 and 3, the method is further explored and became more flexible.

The method can be generalized as a way of thinking based on the known technique of proving colinear relationships by introducing a new point on the line.

Discovery

Problem 1

In triangle ABC, point A_1 lies on side BC and point B_1 lies on side AC. Let P and Q be points on segments AA_1 and BB_1, respectively, such that PQ is parallel to AB. Let P_1 be a point on ray PP_1, such that B_1 lies strictly between P and P_1, and $\angle PP_1C = \angle BAC$. Similarly, let Q_1 be a point on ray QA_1, such that A_1 lies strictly between Q and Q_1, and $\angle CQ_1Q = \angle CBA$.

Prove that points P, Q, P_1, and Q_1 are concyclic. (2019 IMO P2)

Analysis: It is natural to introduce the intersection of the two sides of equal angles, then concyclic relationship follows. Monge theorem tells us that the desired result is equivalent to the fact that the radical axis of the two circles given by the two groups of concyclic points (let us call it l for short) passes through the intersection of PP_1 and QQ_1 (let us call it R). Here comes the central part of the question: how to prove that the intersection is on the radical axis? Well, as point C is already on the radical axis, we only need to find another point on l which is also colinear with C and R. This attempt seems to be impossible since no points on l has a good connection with other parts of the figure, but there is one available point: the intersection of AB and l (let us call it K). The radical axis provides a ratio condition on AB and that can be made use of. To prove such a colinear relationship with parallel and a strange ratio in the configuration, the author was inspired to combine these two and then discovered the following solution.

Solution: Let PP_1 meet QQ_1 at R; BC meet PP_1, QQ_1 at S, T, respectively; AQ meet BP at D; and DR meet AB, PQ at K, L, respectively.

Since A, P, and A_1 are colinear and B, Q, and B_1 are also colinear, Pappus theorem (a case of degenerated Pascal theorem) tells us that C, D and R are colinear.

We are told that $\angle PP_1C = \angle BAC$, so A, S, C, P_1 are concyclic, mark this circle as ω_1. Similarly, B, T, C, Q_1 are concyclic, mark this circle as ω_2.

Because $PQ//AB$, we have $\frac{AK}{BK} = \frac{PL}{QL} = \frac{SK}{TK}$. Hence K lies on the radical axis of ω_1 and ω_2. At the same time, C is also on the radical axis of ω_1 and ω_2 since C is one of their intersections. So, CK is the radical axis of ω_1 and ω_2, then, by power of point R and the fact that $PQ//ST$, we have that $\frac{RP_1}{RQ_1} = \frac{RT}{RS} = \frac{RQ}{RP}$, which indicates that $P,\ Q,\ P_1,\ Q_1$ are concyclic.

Commentary: The thinking is already given in the analysis. As a matter of fact, the author discovered the method when the author was trying to solve this problem. He realized that some hidden patterns exist in the above process. With further exploration and thinking, the writer managed to solve the two following problems.

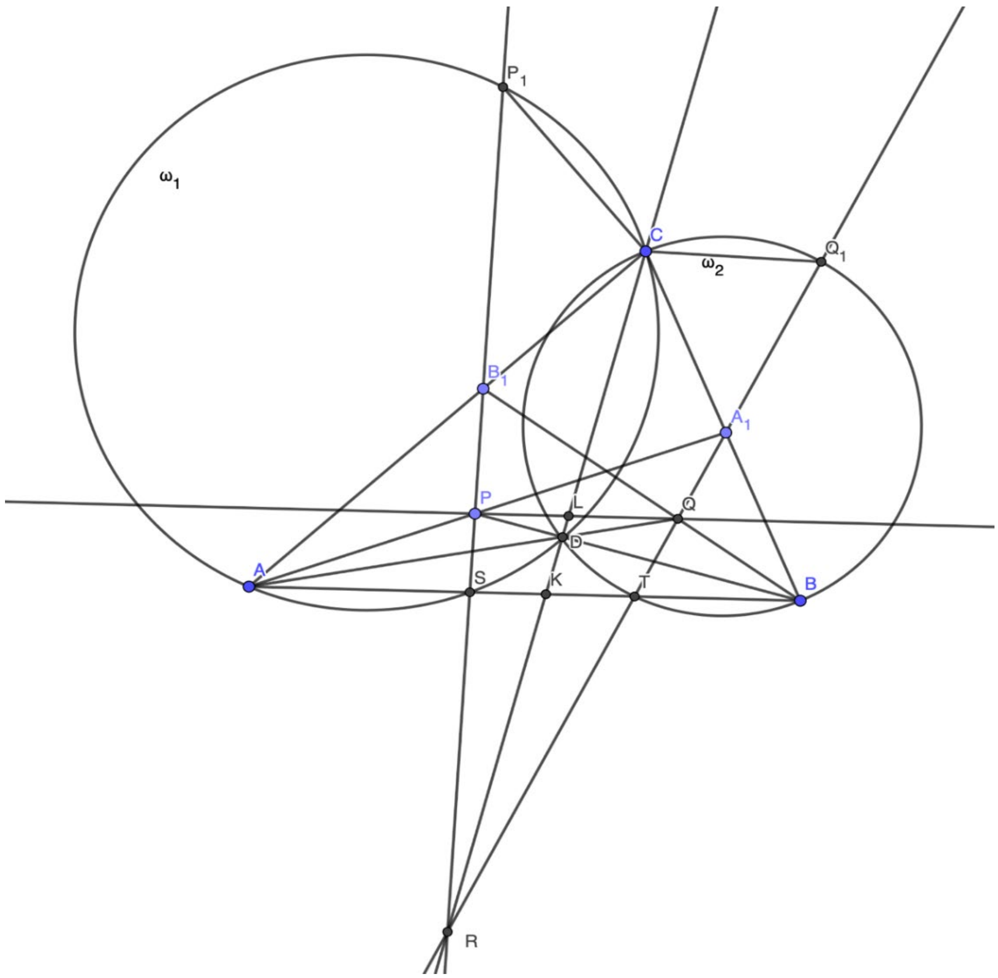

Figure 1. Problem 1

Further Exploration

Problem 2

Point D is selected inside acute triangle ABC so that ∠DAC = ∠ACB and ∠BDC = $\frac{\pi}{2}$ + ∠BAC. Point E is chosen on ray BD so that AE = EC. Let M be the midpoint of BC.Show that line AB is tangent to the circumcircle of triangle BEM. (2024 USAMO P5)

Analysis: To use the unusual angular relationship, a natural attempt is to consider homothety with center at B or C so that the condition can be transformed to a concyclic relationship. Here, the author tried introducing G', a point on CD satisfying that AG' is perpendicular to AC. (To be honest, a quicker, shorter solution can be discovered if the homothety about B is considered.). Then we have that A, D, C, G' are concyclic. Suppose that the result is already proved, then with some angle chasing, more concyclic relationships are discovered but they have poor relationship with the original points. So, the writer cancelled the homothety and tried to directly prove the concyclic relationships (which are equivalent to the desired result). By cancelling the homothety, we get some new points, including K and G below, and parallel conditions appear due to homothety. Then, the configuration turns out to be a familiar one: by directly applying the conclusion mentioned in the introduction part, we can get necessary colinear conditions for angle chasing, therefore prove the concyclic relations and finish the proof.

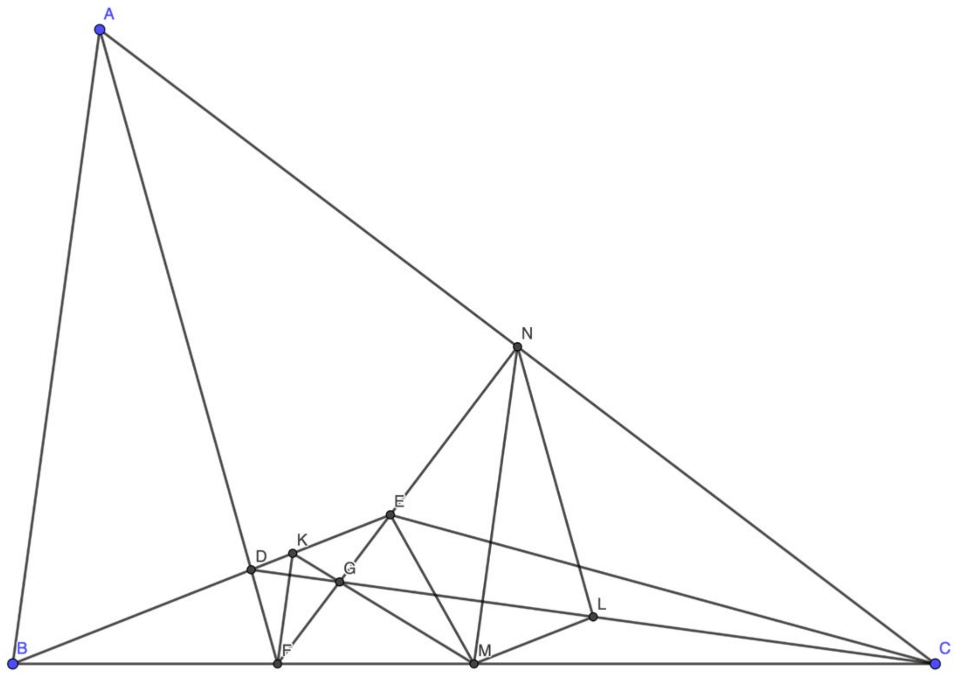

Figure 2. Problem 2

Solution: Let $F = AD \cap BC$, $L = EM \cap CD$. Let L and N be the midpoints of CD and AC, respectively. Let K be a point on BE so that $FK // AB$.

We know that $AF = CF$ as $\angle DAC = \angle BAC$, and because $AE = CE, AN = CN$, E, F, and N all fall on the perpendicular bisector of segment AC.

Due to parallel, $\triangle$ ABD and $\triangle$ FKD are homothetic with inner homothetic center D. Due to midpoints, $\triangle$ ABD and $\triangle$ NML are homothetic with outer homothetic center C. So, $\triangle$ FKD and $\triangle$ NML are homothetic. Let G be their outer homothetic center, then C, D and G are colinear. At the same time, the properties of the homothetic center show that KM and FN both pass through G.

Since $\angle GDK = \pi - \angle BDC = \frac{\pi}{2} - \angle BAC = \angle NFK$, D, F, G, K are concyclic. Since $\angle EKM = \angle DFG = \frac{\pi}{2} - \angle DAC = \frac{\pi}{2} - \angle ACB = \angle CFN$(here the last equality is true as $FK // AB$ and $FN \perp AC$), E, K, F, M are concyclic. So $\angle BME = \angle BKF = \angle ABE$, from which the tangential relationship follows.

Commentary: The deduction using homothetic figures is just another way of using parallel lines' ratio properties, it can somehow shorten the solution but is not necessary at all. The problem is not actually very dependent on the method and has a lot of other simple solutions. But unlike most solutions, the one the writer gives here does not seek for a way to transfer the angle condition to find more geometric properties: the application of the method and the introduction of K and G help to circumvent such process. In fact, this short solution is not so simple. The introduction of point K and point G is a seemingly bold but subtle step based on previous attempts and scrutinizing of the angular relationships in the figure. By the way, from the conclusion, a Miquel point configuration (F is the Miquel point and two of the four circles are already given in the solution) can be discovered with this courageous attempt. To be honest, this solution does not have a strong relationship with the method, but it demonstrates a way how the conclusion and the method can be associated with other geometries and better applied.

Problem 3

In triangle ABC, AB > AC The bisector of ∠BAC meets BC at D. P is on line DA, such that A lies between P and D. PQ is tangent to ⊙ (ABD) at Q. PR is tangent to ⊙ (ACD) at R. CQ meets BR at K. The line parallel to BC and passing through K meets QD, AD, RD at E, L, F, respectively. Prove that EL = FK. (2019 CMO P2)

Analysis: Although the conclusion has nothing to do with line EF but is only about a pencil of lines passing through D (DQ, DR, DA, DK, BC), we do not have enough information about DK, so it does not seem to be a practical attempt. The writer noticed that the parallel relation could bring good ratio qualities since the writer was already familiar with the property in the introduction part. Then, it is natural to guess that ST//BC. With the help of angle chasing, it can be found that S, T, R, Q are concyclic. Then B, C, R, Q must be concyclic, which is the lemma. The lemma does not have any relationship with our theme, so let's ignore it. Then, with the success in problem 1, the author applied the Monge theorem and Pappus theorem, deriving that AD, BT, and CS are concurrent. Which, when combined with the parallel lines, can derive the desired result.

Solution: Note the circumcircle of triangle ABD ω_1, the circumcircle of triangle ACD ω_2.

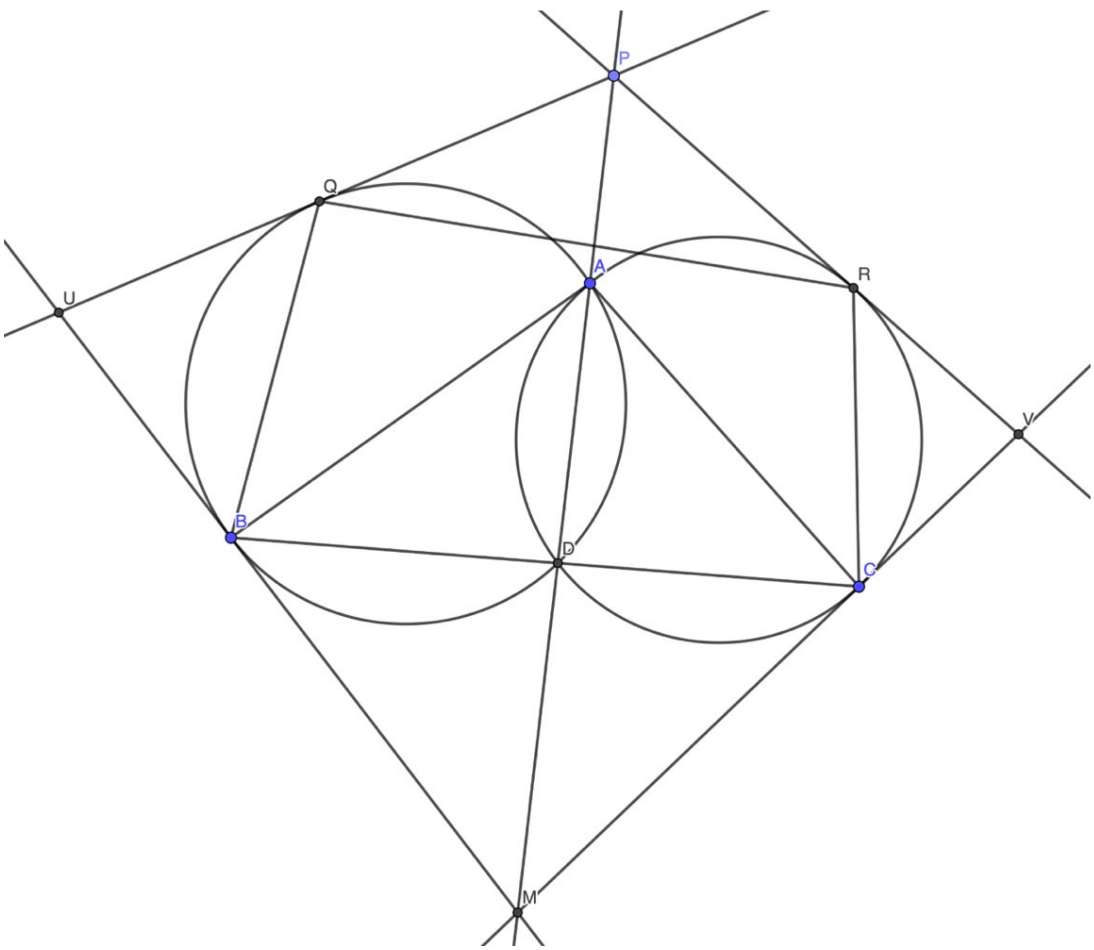

Figure 3. lemma of problem 3

Lemma: B, C, R, Q are concyclic.

Proof: Let $M = AD \cap \odot(ABC)$. It is well-known that MB=MC, and MB is tangent to ω_1. Similarly, MC is tangent to ω_2. Let $U = MB \cap PQ, V = MC \cap PR$. Due to tangent, we have UB=UQ, VC=VR. In addition, $PQ^2 = PA * PD = PR^2$, so PQ =PR. Therefore, $\angle BQR = \pi - \angle BQV - \angle PQR = \frac{1}{2}(\pi - 2\angle BQU + \pi - 2\angle PQR) = \frac{1}{2}(\angle P + \angle U) = \pi - \frac{1}{2}(\angle Q + \angle V)$. Similarly, $\angle BCR = \frac{1}{2}(\angle Q + \angle V)$, so B, C, R, Q are concyclic.

The proof of the lemma is done and let's go back to the question.

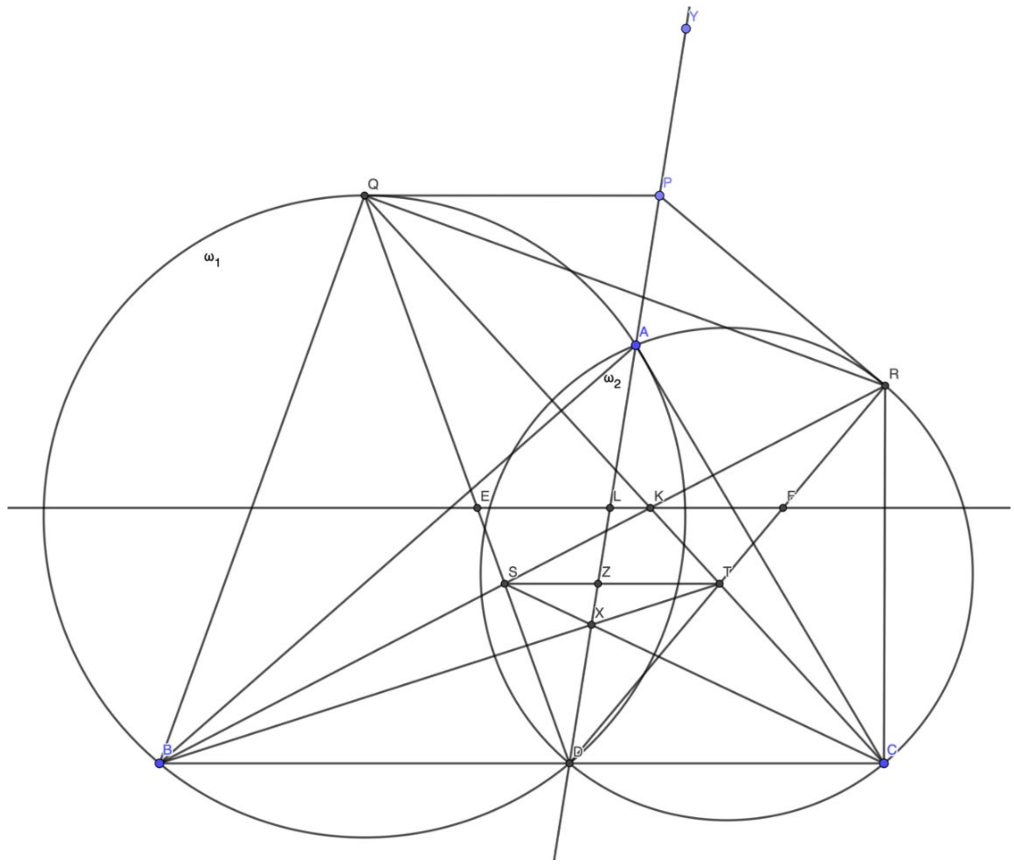

Figure 4. problem 3

Let $S = BR \cap DQ$, $T = CQ \cap DR, X = BT \cap CS, Y = BQ \cap CR, Z = AD \cap ST$ (If $BQ // CR$, let Y be the point at infinity, the following reasoning still stands as all the theorem used also apply for parallel lines). Applying Monge theorem to ω_1, ω_2, and the circumcircle of BCRQ, we have: Y falls on AD (Y is omitted for it is too far away from all the other points) . Since B, S, R are colinear and C, T, Q are also colinear, we know that X, D, Y are colinear by Pappus theorem. So, X also lies on AD.

It is true that S, T, R, Q are concyclic due to the following process of angle chasing: $\angle SQT = \angle BQC - \angle BQD = \angle BRC - \frac{1}{2}\angle BAC = \angle BRC - \angle CRD = \angle SRT$. Hence $\angle STQ = \angle SRQ = \angle BCQ$, which implies that $ST//BC$.

We will finish the proof via some ratio chasing:

On one hand, $\frac{EL}{FL} = \frac{SZ}{TZ} = \frac{CD}{BD}$; On the other hand, $\frac{CD}{FK} = \frac{CT}{TK} = \frac{CS}{SK} = \frac{BD}{EK}$, therefore $\frac{FK}{EK} = \frac{CD}{BD} = \frac{EL}{FL}$, which leads directly to the conclusion that $EL = FK$.

Commentary: The proposal of the lemma is natural according to both speculating and calculating (in fact, the proof can be done with the lemma and acceptable amount of calculation). By discovering and further exploring the parallel relation between ST and BC, we can complete the process of the ratio chasing in the end of the solution. Fortunately, the colinear relationship we need for the ratio chasing (X falls on AD) can be dealt with point Y as a medium using the method. Such, the proof is done.

Conclusion

To sum it up, a general description of the method is given below: when we want to prove a colinear relationship, there is a common method to introduce auxiliary points and combine the use of various theorems to prove the desired result. In the author's method, a special connection between certain theorems is noticed and applied. For instance, the Desargues theorem and the Pappus theorem both include intersections of lines with some other colinear relationships, therefore, they can be used together. Besides, when parallel lines appear, special cases and degenerated forms of theorems appear, this fact brings extra properties: like in problem 1, point R lies on the radical axis. Hence, the conclusion proposed in the introduction part is useful since on many occasions, the degenerated cases of theorems are easily neglected. The examples above discussed the situation for Pappus theorem, Monge theorem and a degenerated case of Desargues theorem (the conclusion mentioned in the introduction, which is also a special case of a property of homothetic centers).

Limitations and Foresights

Due to the limited level of the author, there may be more efficient methods of proving colinear relationships not mentioned since the essay only mentioned a few theorems. For example, Pascal theorem can be somehow jointed, and even conclusions in projective geometry can be used in the process, which would bring a larger variety of methods available when trying to prove colinear relationships. Besides, the connection between theorems about colinear points may have other means of connecting with other geometric properties and objects. Other theorems like Menelaus theorem can also be introduced.

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artofproblemsolving.com