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Understanding Capillary Rise Dynamics: A Theoretical Investigation

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ABSTRACT

Capillary tubes are very thin tubes made of a rigid material, such as plastic or glass in which a liquid flows up into the tubes against gravity in a process called capillary action. Capillary action occurs due to the liquid's surface tension and intermolecular forces without the assistance of external forces or any other medium of energy. It happens because of the intermolecular forces between the liquid and its surroundings. The diameter of the tube plays a crucial role in this process. If the diameter of the tube is sufficiently small, the adhesive forces between the liquid and the glass of the tube act to propel the liquid upward. This theoretical framework of my research incorporates the forces of gravity, surface tension, and viscosity to find an expression for the time liquid takes to reach a specific height in a capillary tube.

Introduction

Capillary rise, the phenomenon where a liquid ascends in a narrow tube due to intermolecular forces, has fascinated scientists for centuries. From its early observations by Leonardo da Vinci to its modern-day applications in various fields such as microfluidics, biology, and material science, the dynamics of capillary rise continue to intrigue researchers worldwide. In this paper, I have worked on the theoretical analysis of the time taken by a liquid to ascend in a capillary tube, elucidating the underlying principles governing this process.

The capillary rise phenomenon arises from the delicate balance between cohesive forces within the liquid and adhesive forces between the liquid and the solid surface of the tube. Surface tension, viscosity, tube geometry, and gravitational effects all play crucial roles in determining the rate at which the liquid ascends.

My research begins with a comprehensive examination of the fundamental principles governing capillary rise dynamics. I derive equations describing the change in pressure within the capillary tube, considering factors such as surface tension, gravitational forces, and fluid viscosity. Through rigorous mathematical analysis, I have developed a theoretical framework to predict the time required for the liquid to ascend a given height in the capillary tube.

Furthermore, I explore the implications of my theoretical findings and their relevance to practical applications. By validating my theoretical model through comparison with experimental data and established empirical relations, I aim to provide valuable insights into the underlying mechanisms driving capillary rise phenomena.

In summary, this research paper presents a detailed theoretical investigation into capillary rise dynamics, shedding light on the fundamental principles governing this phenomenon. By combining theoretical analysis with practical implications, we offer a comprehensive understanding of capillary rise dynamics, paving the way for advancements in various fields reliant on this phenomenon.

Literary Framework

Capillary action has been an interesting topic of research in physics for centuries. It began with Leonardo da Vinci's observation of water rising in narrow tubes and Galileo Galilei's study of capillary tubes. Later, researchers such as Thomas Young and Jurin introduced concepts of angle of contact and capillary height, further contributing to the study of the phenomenon.

In the modern world, capillary action has practical applications in various fields such as physics, biology, chemistry, and medicine. Microfluidics technology has revolutionized the research and study of capillary action, enabling precise control of fluid at a molecular level. Recent advancements have led to the development of capillary drug delivery in patients, environmental sensors, and other innovative applications.

Despite centuries of intensive research, there are still many mysteries in this topic, and many questions remain unanswered. Theoretical models and studies continue to refine our understanding of fluid dynamics, paving the way for new and innovative research and conclusions.

Parameters

1. Surface Tension →S
2. Coefficient of Viscosity →μ
3. Pressure →P = $\frac{Perpendicular\ component\ of\ force}{Area}$
4. The angle of contact → θ
5. Density of liquid →ρ
6. Acceleration due to gravity →g
7. The radius of the capillary tube→R
8. Length of capillary tube →L
9. Time taken by liquid to reach height x →T
10. Height of liquid in capillary tube →x
11. Atmospheric Pressure→ P_{atm}
12. Force due to surface tension→ Product of surface tension with length on which the force acts

Assumptions

1. The volume of liquid in the meniscus, which is the curved surface formed due to the forces of surface tension, is negligible in our calculations. This is because the volume of liquid in the meniscus is very small compared to the total volume of liquid in the container. The meniscus is formed due to the interaction between the liquid and the walls of the capillary tube, which causes the surface of the liquid to curve upwards.
2. I have assumed that the viscous force between the wall of the capillary tube and the liquid is zero, meaning that there is no resistance to the flow of the liquid along the walls of the tube. (The thickness of the boundary layer is to be infinitesimally small)
3. To simplify calculations, I have assumed that the length of the capillary tube inside the liquid container is minimal. This is required because the length of the capillary tube inside the container will have a negligible effect on the overall liquid flow.
4. I have only considered a laminar flow of liquid (the liquid flows smoothly and predictably without any turbulence or chaotic movement.), characterized by a Reynolds Number of less than 1000. The Reynolds Number is a dimensionless quantity that describes the ratio of inertial forces to viscous forces in a fluid. A Reynolds Number of less than 1000 indicates that the flow of liquid is smooth and

predictable, with no turbulence or chaotic movement. This means that the liquid flows in parallel layers, and there is no mixing or swirling of the liquid (Newtonian Fluid).

5. I assume axisymmetric flow within the capillary tube, where the velocity and other flow characteristics are independent of the azimuthal angle around the tube's central axis. This assumption simplifies the mathematical treatment by reducing the problem to one dimension, assuming that variations in flow properties occur only along the radial direction.
6. I assume isothermal conditions within the capillary tube, meaning that the temperature remains constant throughout the fluid. This simplifies the analysis by neglecting heat transfer effects on the liquid flow.
7. I assume axisymmetric flow within the capillary tube, where the velocity and other flow characteristics are independent of the azimuthal angle around the tube's central axis. This causes the problem to remain up to one dimension only, (assuming that variations in flow properties occur only along the radial direction)

When we conduct an analysis, we often rely on assumptions to simplify the process and make it more manageable. However, we must also consider the practicality of these assumptions in real life scenarios and their relevance to the specific situation under investigation. Therefore, it is crucial to validate the results with experimental findings to ensure that they accurately reflect the situation at hand.

The Proof

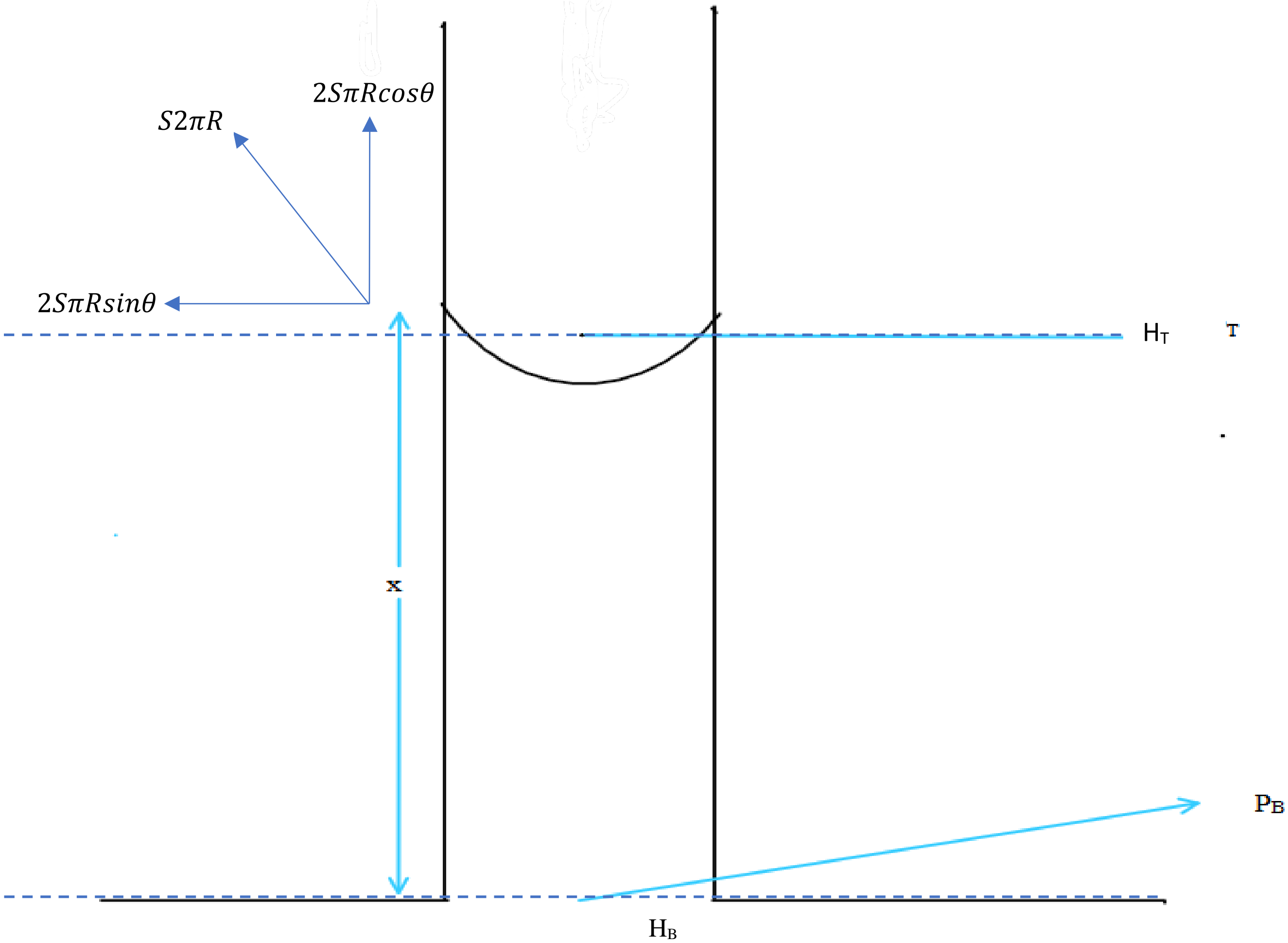

Figure1. Condition of liquid in the capillary tube at time T

Let at any time T, the liquid reach a height x in the capillary tube.

Let the pressure at the top of the surface of the liquid be P_T and that at the bottom of the capillary tube be P_B. Considering the bottom of the capillary tube as a reference:

$$P_T=P_{atm}-\frac{2Scos\theta\pi R}{\pi R^2}$$
$$P_T=P_{atm}-\frac{2Scos\theta}{R}....(1.1)$$

(Only the cosine component of the force of surface tension acts in the perpendicular direction. Therefore, we can write pressure due to surface tension as $\frac{2Scos\theta}{R}$)

$$P_B=P_{atm}.....(1.2).$$

Therefore, the change in pressure (ΔP) can be written as:

$$\Delta P=P_B-P_T$$
$$\Delta P= (P_{atm}+\rho gH_B) -(P_{atm}-\frac{2Scos\theta}{R}+\rho gH_T)$$

$\Delta P = \frac{2S\cos\theta}{R} + \rho g(H_B - H_T)$

H_B and H_T are the liquid column heights at the bottom and top, respectively.

Considering H_B=0 and H_T=x (Taking the bottom of the capillary tube as a reference), we get:

$$\Delta P = \frac{2S\cos\theta}{R} - \rho g x \qquad ......(1.3)$$

Now, using Poiseuille's equation:
$\Delta P = \frac{8\mu LQ}{\pi R^4}$

Here, L=x for the given condition, and Q=Volume flow rate= Area x Velocity. Now, the velocity at this instant can be written as the derivative of the change in height concerning time (t).

$\Delta P = \frac{8\mu x \pi R^2 \frac{dx}{dt}}{\pi R^4}$

Simplifying and using equation 1.3, we get:

$$\frac{\frac{2S\cos\theta}{R} - \rho g x}{x} = \frac{8\mu \frac{dx}{dt}}{R^2}$$

$$\frac{2S\cos\theta}{Rx} - \rho g = \frac{8\mu \frac{dx}{dt}}{R^2} \qquad .....(1.4)$$

Further simplifying:

$$\frac{2SR\cos\theta}{8\mu x} - \frac{\rho g R^2}{8\mu} = \frac{dx}{dt}$$

Let $\frac{2SR\cos\theta}{8\mu}$=a and $\frac{\rho g R^2}{8\mu} = b$

$$\frac{a}{x} - b = \frac{dx}{dt}$$

$$dt = \frac{dx}{\frac{a}{x} - b}$$

Integrating both sides, we get:

$\int_0^T dt = \int_0^x \frac{x.dx}{a - bx}$.......(1.5)

Let a-bx=t
Taking derivative both sides we get:

$$\frac{-dt}{b} = dx$$

Also,

$$x = \frac{a-t}{b}$$

Putting these two equations in equation 1.5:

$$T = \int \frac{a-t}{bt} dt$$

$$T = \frac{1}{b^2}[t - a \ln|t|]$$

Putting in the limits of the integral from 0 to x and the values of a and b:

$$T = \left[\frac{-1}{b^2}(a \ln(|a - bx|) + bx - a \ln a)\right]_0^x$$

$$T = \frac{-64\mu^2}{\rho^2 g^2 R^4}\left(\frac{2Scos\theta R}{8\mu}\left(\ln \left|\frac{2Scos\theta R - \rho g R^2 x}{8\mu}\right|\right) + \frac{\rho g R^2 x}{8\mu} - \frac{2Scos\theta R}{8\mu}\left(\ln \frac{2Scos\theta R}{8\mu}\right)\right)$$

Validation and Verification

To verify the given equation, we need to substitute certain values for the variables and check if the equation holds.

When we substitute x=0, we get T=0. This means that when the value of x is zero, the value of T is also zero. This is an expected result and satisfies one of our boundary conditions, confirming that the model behaves correctly at the initial point where the liquid begins to rise in the capillary tube.

Next, we substitute $H_{max}=\frac{2Scos\theta}{\rho g R}$ (Appendix) in the equation. When we do so, we get T→∞. This shows that it takes infinitely long time for the liquid to reach maximum height in the capillary tube.

By demonstrating that the equation satisfies the conditions of zero time at the capillary base and infinite time at maximum height, I validate that my equation accurately models the capillary rise phenomenon. This verification process supports the reliability and robustness of my theoretical model, ensuring that it correctly describes the behavior of liquid rise in a capillary tube under the specified conditions.

Experimental Graph

When we put in the values of the parameters for water, those are:

g=9.8 meters per second square
$\rho=10^3$ kilograms per meter cube
$\mu=10^{-3}$ Pascal-second
$R=10^{-3}$ meter or 1 millimetres
S=72.8 millinewtons
$\theta = \frac{12\pi}{180}$

in the function of time, we obtain the following graph for the function:

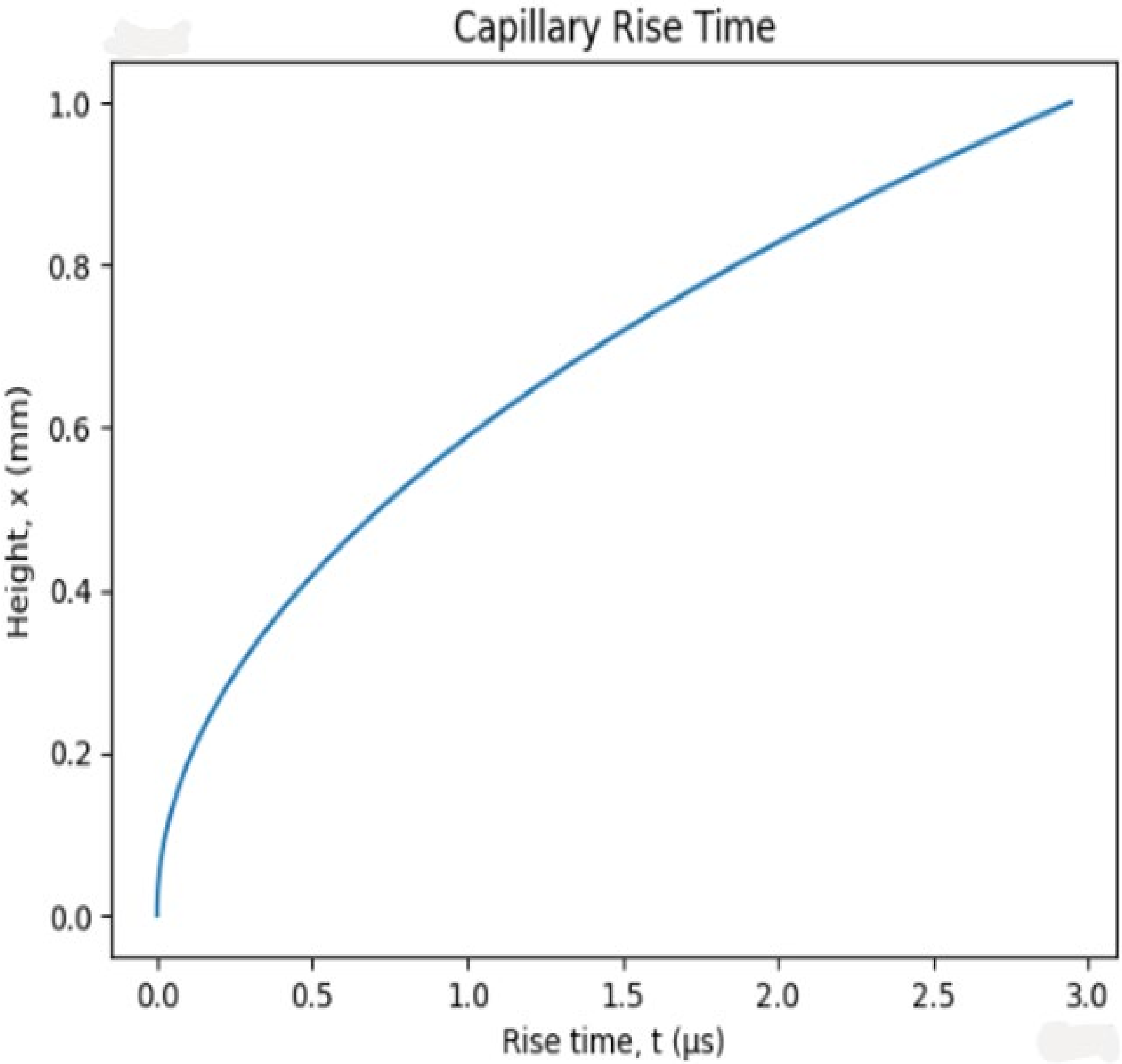

Figure.2. Capillary rise time graph for water

The X-axis represents the Height reached in millimeters (10^{-3} meters)
The Y-axis represents time taken in μ-seconds (10^{-6} seconds)

The graph provides a visual representation of the height of the capillary tube's water column as a time function. Initially, the height increases rapidly, but as time progresses, the rate of rise slows down, approaching a horizontal asymptote. This asymptotic behavior indicates that the liquid takes an infinitely long time to reach its theoretical maximum height in the capillary tube.

This behavior is consistent with the theoretical predictions of the capillary rise phenomenon, where the driving force (surface tension) and the resistive forces (gravitational and viscous forces) reach a dynamic equilibrium.

The following graph represents experimental data from previous research.

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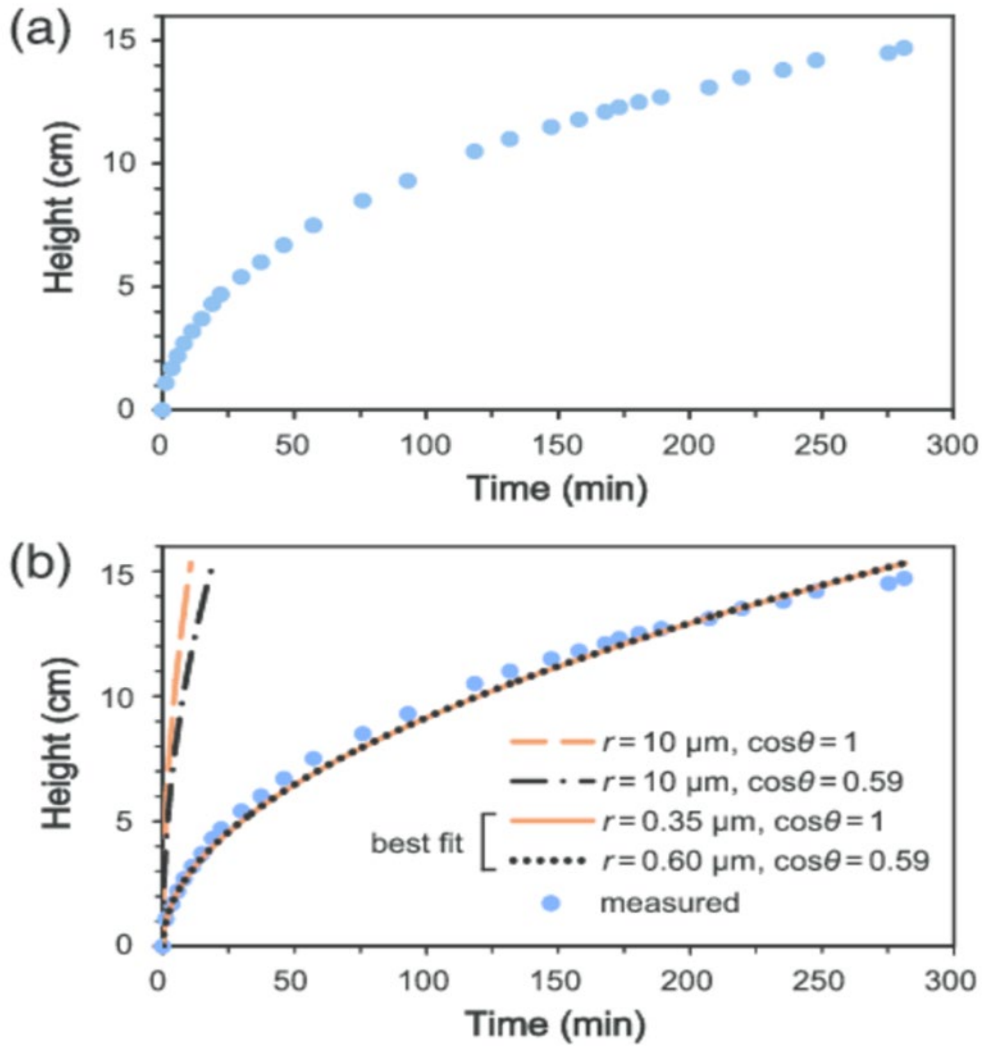

Figure 3. Capillary rise time graph from previous experimental data. (Source: https://www.researchgate.net/figure/Measured-and-calculated-values-of-capillary-rise-over-time-Examples-of-capillary-rise_fig4_301714732)

When comparing Fig.2 (my graph) with Fig.3 (previous experimental data), several key observations can be made:

Similar Asymptotic Behavior: Both graphs exhibit a similar asymptotic trend, indicating that the height of the liquid column increases rapidly initially and then gradually slows down, approaching a maximum height asymptotically.

Rate of Rise: The rate at which the height increases over time is comparable in both graphs, further validating our model.

Consistency in Parameters: The use of similar physical parameters (density, viscosity, surface tension, etc.) in our model and the previous experimental setup reinforces the reliability of our results.

Result

The final result found using analysis of my equation and its graph is that it takes infinite time for the liquid to rise in a capillary tube and depends upon various parameters such as surface tension, tube geometry, and gravity. The ascent time can be controlled in practical applications by manipulating these parameters. Additionally, the limitations of this theoretical model can be potential avenues for future research.

Conclusion

In conclusion, my research theoretically analyzes capillary rise dynamics in a thin capillary tube. By developing a theoretical model to predict the time liquid takes to rise in a capillary tube, I contribute to the fundamental understanding of this phenomenon. The validation of this model with experiments enhances its accuracy. My research offers insight into the design and optimization of capillaries in scientific and technological applications.

Acknowledgment

I thank all my teachers, advisors, friends, and parents for providing me with positive insight about my topic and supporting me.

Appendix

To verify my final equation, I have taken $H_{max}=\frac{2Scos\theta}{\rho gR}$.

The proof of this can be done this way:

When the liquid reaches its maximum height, ΔP will be equal to zero (Because velocity will be zero).

Therefore,

$$\frac{2Scos\theta}{R} = \rho g H_{max}$$

$$H_{max} = \frac{2Scos\theta}{\rho gR}$$

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