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A Review of the Existence of Space-Time Singularities

Mihir Bhatlawande[1] and Shilpa Ghadge[#]

[1]The Orchid School, India
[#]Advisor

ABSTRACT

Ever since the discovery of the Schwarzschild metric the structure of the interiors of black holes has been one of the biggest questions of gravitational physics. The Raychaudhari theorem proposed a possible singularity at the center, and the Penrose Singularity theorem claimed to prove the existence of the singularity at the center of black holes. This was taken to be fact by the community until recently when a paper by physicist Roy Kerr laid down a structured argument against singularities. We analyze both the argument for as well as against singularities. It was seen that what the Penrose singularity theorem proved was not the existence of singularities. Further research and new physics are required to determine whether singularities exist.

Introduction

The Schwarzschild solution was the first solution to Einstein's field equations. It represents the gravitational field outside any spherically symmetric mass. What was puzzling was that the metric seemed to be singular at two regions: r=0 and r=2m (we will see that the latter was not a real singularity). The idea of the black hole started to emerge from the works of Schwarzschild, Oppenheimer and Snyder, etc. Almost 40 years after Einstein put forward General Relativity, the world of relativistic physics was introduced to the first-ever singularity theorem in the form of Raychaudhari's paper. This paper was followed not long after by Roger Penrose's paper which provided seemingly a more conclusive proof of singularities inside black holes in General relativity. Almost 60 years after the publication of Penrose's paper, reputed physicist Roy Kerr put forward his own ideas breaking down the assumptions and methods used in the "proofs" of singularity theorems. We will begin our analysis with the Singularity theorems and then move forward to the recent Kerr paper.

The Raychaudhari Theorem

Any discussion of spacetime singularities must include the Raychaudhari theorem.

Raychaudhari's Theorem: Assuming $\Lambda = 0$, and an energy-momentum tensor that is a perfect-fluid

$$T_{\mu\nu} = \varrho u_\mu u_\nu + p(g_{\mu\nu} + u_\mu u_\nu), \qquad u^\mu u_\mu = -1$$

(1)

whose velocity vector field u^μ is geodesic and irrotational. "If the expansion is positive (resp. negative) at an instant of time and the convergence condition holds, then the energy density ϱ of the fluid diverges in the finite past (future) along every integral curve of u^μ" (Senovilla & Garfinkle, 2015).

The Raychaudhari theorem concerns the "kinematics of flows" (Senovilla & Garfinkle, 2015) of geodesic vector fields which have the property that they are non-accelerated and irrotational. The condition

$$R_{\rho\nu}u^{\rho}u^{\nu} \geq 0$$
(2)

is called the null convergence condition. This condition is necessary for the "focusing effect" on the null geodesics (as well as on all other causal geodesics). The expansion of the flow of the vector field,

$$\theta \equiv \nabla_{\mu}u^{\mu} = S^{\mu}{}_{\mu}$$
(3)

must be negative (for black holes. Resp, positive for white holes). This represents the contracting(Resp, expanding for white holes) null vector flow, and thus spacetime. The expansion is a component of the gradient of the velocity vector field. It can be illustrated as follows:

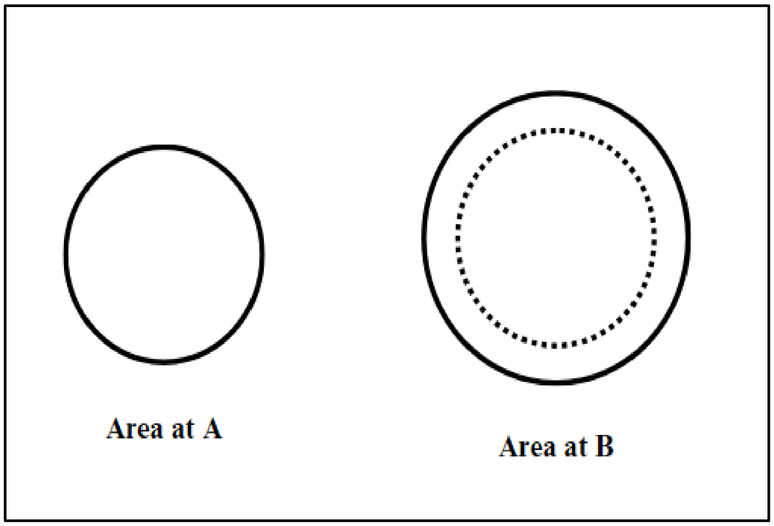

Figure 1. Illustrating expansion. When $\theta < 0$ contraction occurs, and when $\theta > 0$ expansion occurs.

For an in-depth analysis of the Raychaudhari equations, one may consult the paper by (Kar, S., Sengupta, S. et. al.). However, for the purpose of our discussion, we will cover the result of the theorem. When the above conditions are satisfied, we find that the energy density of the fluid diverges in a finite past or future. In other words, the energy density at a point along all the geodesic curves tends to infinity.

Although this may imply a singular region in the spacetime manifold, it is not conclusive for the existence of singularities. After this influential paper, we soon saw the publication of the paper titled, "Gravitational Collapse and Space-Time Singularities" by Sir Roger Penrose. This paper sought to deliver a more general and conclusive theoretical proof for singularities inside black holes.

Penrose Singularity Theorem

The paper by Penrose aimed to prove that singularities are inevitable regardless of whether spherical symmetry was present. The new and revolutionary ideas that Penrose introduced were (i) geodesic incompleteness to describe singularities, and (ii) closed trapped surfaces.

Geodesic Incompleteness

A singularity in a Lorentzian manifold is an incomplete endless curve (Senovilla & Garfinkle, 2015). In general relativity, a spacetime's shape is measured using Geodesic curves. In particular, null geodesic curves are used. Null geodesics are curves for whom the spacetime interval $ds = 0$. In more intuitive terms, it is the path that is traveled by a massless particle.

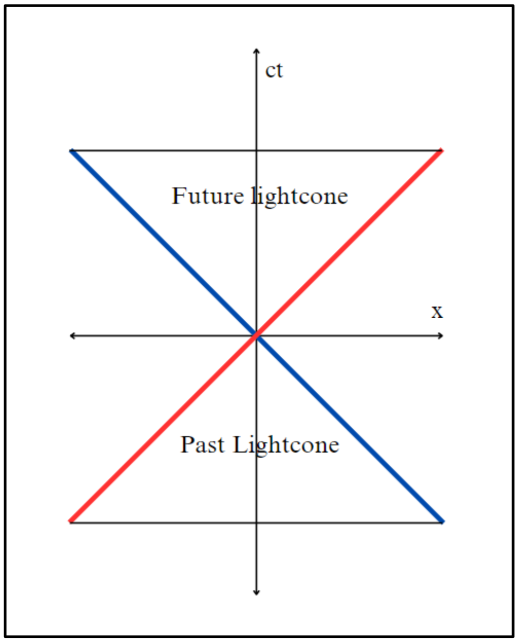

Figure 2. Lightcone in a two-dimensional Minkowski spacetime. The red line represents the path of a light ray moving forward in time and the blue line represents the path of a light ray moving backward in time.

The shape and curvature of the shape of spacetime are measured by the paths that massless particles take in it. When such paths terminate at a finite parameter like time, radius, etc., it must mean that a spacetime singularity is present (meaning that spacetime itself terminates at that parameter) according to Penrose. However, in later sections, we will cover a flaw in this approach.

Trapped Surfaces

By definition, a closed trapped surface is "a two-dimensional imbedded submanifold S (surface), compact without boundary (closed), such that the two families of light rays emerging orthogonally from S towards the future converge initially (trapped)" (Senovilla & Garfinkle, 2015). Mathematically this can be defined by

$$\theta_+ < 0, \qquad \theta_- < 0 \tag{4}$$

where,

$$\theta_\pm = \nabla_\mu k_\pm^\mu$$

One can apply a test to understand closed surfaces to check whether a surface is a closed trapped surface. If you are inside the volume contained within the surface, you cannot get out; if you are outside the volume, you cannot get in – without crossing through the surface.
In (4), we consider a surface S, and the two families of light rays emerging orthogonally from S, $k^{\mu}_{\pm}$. If the spacetime is contracting, the variation in the area of S will be measured by the expansions of the null vector fields($k^{\mu}_{\pm}$), $\theta_{\pm}$.

For a black hole, both the incoming and the outgoing light rays will have a negative expansion. Meaning that as time passes, their areas will shrink. Both the vector fields will be inwards pointing. This can be visualized as follows,

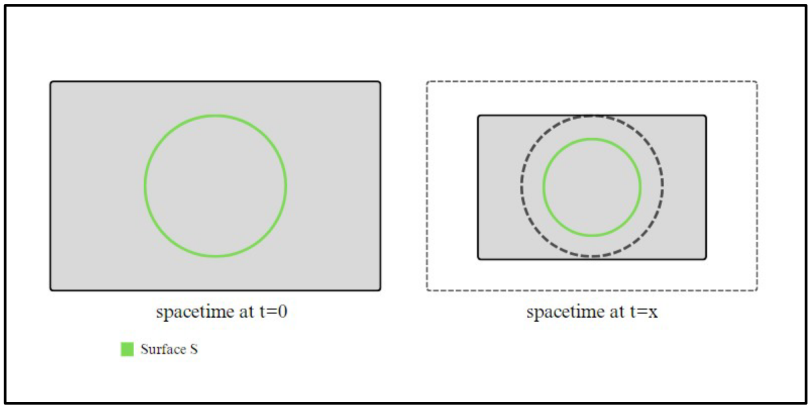

Figure 3. Contracting spacetime. In a closed future trapped surface, the areas of both the outgoing and ingoing light rays will be less than the area of the surface they were emitted from.

We now move on to the Penrose Singularity theorem.

Penrose's Theorem: "If the space-time contains a non-compact Cauchy hypersurface Σ and a closed future-trapped surface, and if the convergence condition holds for null u^{μ} , then there are future incomplete null geodesics." (Senovilla & Garfinkle, 2015).

Notice that the theorem supposedly proves null geodesic incompleteness. We will see in later sections how this is not entirely true. Regardless, for the proof of the theorem one can consult the original paper (Penrose, 1965).

The proof of the theorem is based on a series of assumptions that are inconsistent together. These are:

(i) The spacetime is a nonsingular Riemannian manifold for which the null half cones form two separate systems (those being "past" and "future", refer to Figure 2)

(ii) (ii) Every null geodesic can be extended to an arbitrarily large affine parameter(null completeness)

(iii) (iii) Every timelike or null curve meets the Cauchy hypersurface once

(iv) (iv) There exists non-negative energy density at every point of spacetime

(v) (v) There exists a future closed trapped surface

(vi) Through a proof by contradiction, we arrive at a few possibilities. It may be that the convergence condition is violated for null geodesics, that the concept of space-time breaks down at such conditions, or that some null geodesics are incomplete.

After this theorem, many other similar theorems came up. However, almost all merely prove the existence of Finite Affine Length Light (FALL). We will conclude this half of the argument at the Penrose singularity theorem as the points to be addressed in later sections are common to all such theorems.

Roy Kerr's Argument

In his paper, Kerr addresses whether Hawking and Penrose proved the existence of singularities. According to his argument, relativists have confused affine distance with geodesic distance. This has led them to believe the Penrose Theorem proves that bounded affine parameters lead to singularities.

Limitations of Affine Parameters

Let us consider Geodesic parameters, defined by a first-order differential equation.

$$\frac{ds}{dt} = \sqrt{g_{\mu\nu}\frac{dx^{\mu}}{dt}\frac{dx^{\nu}}{dt}}, \qquad \longrightarrow \qquad s = s_0 + C, \qquad \text{(Kerr, arXiv, 2023)}$$

(5)

This cannot work for null vectors for whom $ds = 0$. Since the proper time parameter for nulls is zero, it cannot be used to track the progress sensibly. We, therefore, replace it with 'affine distance', a(t), which is defined by a second-order differential equation,

$$\frac{d^2x^{\mu}}{dt^2} + \Gamma^{\mu}_{\alpha\beta}\frac{dx^{\alpha}}{dt}\frac{dx^{\beta}}{dt} = \lambda(t)\frac{dx^{\mu}}{dt}. \qquad \text{(Kerr, arXiv, 2023)}$$

(6)

Consulting, Kerr's calculations we get the solution for α as,

$$\alpha = A\alpha_0 + B,$$

where A and B are constants.
The difference between the parameters as mentioned by Kerr is that if λ is constant in (1), $\alpha_0 = e^{\lambda t}$
This implies that a(t) is bounded at either +∞ or -∞.

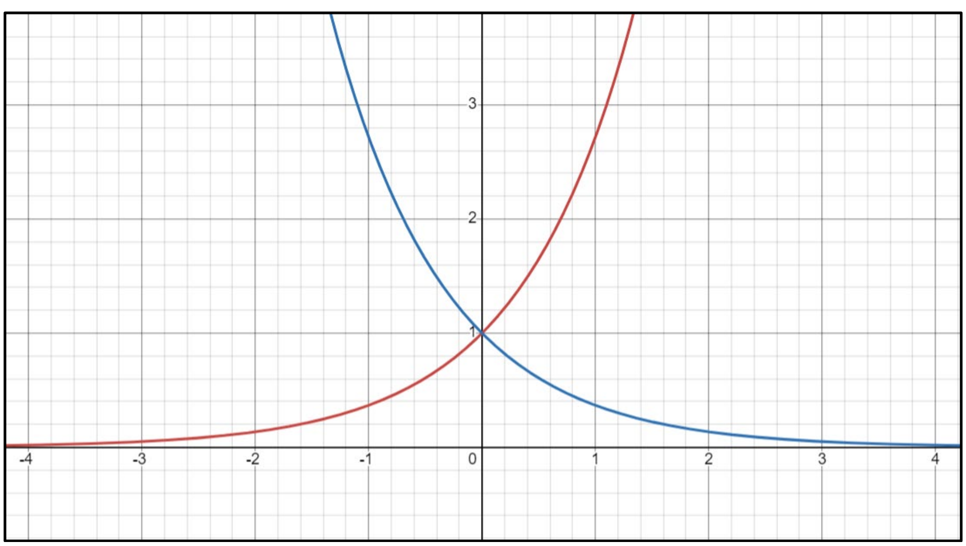

Figure 4. The graphs of $f(x) = e^{x}$ (red) and $f(x) = e^{-x}$ (blue) tend to 0, as $x \to \pm\infty$.

By this, Kerr tries to show that affine parameters being bounded does not imply spacetime being singular. As in the above case, they are bounded without any indication of the curvature of spacetime.

Next in the paper, the author takes a killing vector, k^{μ}, associated with coordinate t,

$$k_{\mu;\nu} + k_{\nu;\mu} = 0, \qquad k^{\mu}\partial_{\mu} = \partial_t,$$

A killing vector field represents a symmetry of the spacetime metric. $k^{\mu}\partial_{\mu} = \partial_t$ shows that k^{μ} has a component along with the coordinate direction t. Therefore, k^{μ} is a time-like Killing vector field. This means that the integral curves of k^{μ} correspond to curves parameterized by coordinate time t.

Following the calculations in Kerr's paper, on multiplying by k^{ν},

$$k^{\nu}k_{\nu;\mu} = 0 \qquad \longrightarrow \qquad k_{\mu;\nu}k^{\nu} = 0,$$

$k_{\mu;\nu}k^{\nu} = 0$ implies that the integral curves of kμ are also geodesics. So we have the result that k^{μ} are geodesics that are parameterized by t and, quoting Kerr, the t-parameter or any affine function of it is affine.

The author uses this result in further sections. He shows that the normals to the event horizons of Kerr and Schwarzschild are light rays of such characteristics. These are principle null vectors (PNVs) with the characteristic that the affine parameters along them are exponential functions of the time parameter t–as discussed above.

$$a(t) = Ae^{Bt}, \tag{7}$$

where A and B are non-zero constants. This vanishes at one or the other end. According to the author, this (affine boundedness) has nothing to do with singularities.

Next, we will assess the metrics individually.

The Schwarzschild Metric

The Schwarzschild metric was the first solution to Einstein's field equations. It describes the gravitational field outside a perfectly spherical mass where the charge, angular momentum, and cosmological constant are zero. The line element is

$$ds^2 = (1 - \frac{r_s}{r})c^2dt^2 - (1 - \frac{r_s}{r})^{-1}dr^2 - r^2d\Omega^2, \tag{8}$$

where $d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$, and $r_s = \frac{2GM}{c^2}$

According to this metric, there are two singularities, one where r=0 (as seen in: $1 - \frac{r_s}{r}$) and the other at the event horizon where r=2m (as seen in: $(1 - \frac{r_s}{r})^{-1}$). However, Eddington and Finklestein, using a different coordinate system, showed that the singularity at the event horizon was not a true one.

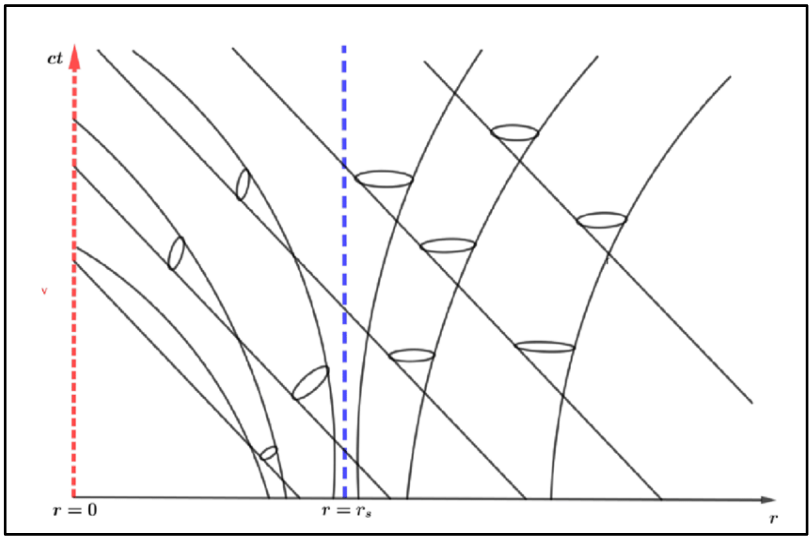

Figure 5. Incoming and outgoing light rays plotted in (advanced) Eddington-Finkelstein coordinates. (Kogut, 2019)

Eddington takes two forms: E- for the black hole, and E+ for the white hole. The time coordinates for the two forms are related to Swarzschild time as,

$$t_- = t_S - 2mln|r - 2m|,\ t_+ = t_S + 2mln|r - 2m|$$

(9)

On subtracting we get,

$$t_+ = t_- + 4mln|r - 2m|,$$

(10)

where t_- is the time coordinate for the black hole and t_+ is the time coordinate for the white hole. Both are related to Schwarzschild time t_S by (10). The line element for S in Eddington-Finkelstein coordinates is

$$ds^2 = (1 - \frac{r_s}{r})dt_\pm{}^2 \quad - \frac{2r_s}{r}dt_\pm dr - (1 + \frac{r_s}{r})dr^2 - r^2[d\theta^2 + (sin\theta)^2 d\phi^2]$$

(11)

Kerr writes these in the Kerr-Schild form as follows,

$$ds^2 = ds_{0\pm}^2 + \frac{2m}{r}(k_{\pm\mu}dx^\mu)^2, \qquad ds_{0\pm}^2 = dr^2 + r^2 d\sigma^2 - dt_\pm{}^2 \quad .$$

$$k_\pm = k_{\pm\mu}dx^\mu = \pm dr - dt_\pm, \qquad K_\pm = K_\pm^\mu \partial_\mu = \pm\partial_r + \partial_{t\pm}$$

Here $K_\pm$ are the family of light rays orthogonal to the event horizon. These are principle null vectors, and we can make a calculation using the fact that the second PNV, $K_\pm^*$, is the other in a different coordinate system, $K_\mp$.It is well known that K_- and K_-^* are inwards pointing inside the event horizon for a black hole and outside the event horizon, K_- points inwards whereas K_-^* points outwards. This can be seen in Figure 3. Both K_-(yellow) and K_-^*(blue) are inwards pointing and their expansions $\theta_\pm < 0$.

We see that these rays are the principal null vectors of the metric tensor which are geodesic and shear-free. As we have seen, the rays do not cross the event horizon in the original Swarzschild coordinates. Yet, K_-

crosses the horizon in E_- and K_-^* is "asymptotic to it as $t_- \longrightarrow -\infty$" (Kerr, arXiv, 2023), meaning that it approaches the horizon closely.

As we have seen in (7) the affine parameter r is bounded as the principal null vectors approach the horizon. According to the author, what has been considered proof of the singularity theorem is far from showing that singularities exist. Affine parameters of such PNVs are bounded with no relation to the curvature of spacetime in the metric. The author also addresses the Kruskal-Szekeres extension to Schwarzschild.

Kruskal Extension

The Kruskal-Szekeres coordinates provide a complete description of Schwarzschild spacetime, inside and outside the event horizon. The original approach to the Kruskal extension begins with the Swarzschild coordinates and then a singular transformation is used to obtain the Kruskal coordinates. Kerr shows that Kruskal is an analytic extension of Eddington and not Schwarzschild. He begins with Eddington (using units 2m=1 for convenience),

$$ds^2 = dr^2 - dt^2 + \frac{1}{r}(dr + dt)^2 = (dr + dt)(dr - dt + \frac{1}{r}(dr + dt))$$

and obtains the Kruskal Szekeres metric (1 is again substituted by 2m).

$$ds^2 = \frac{32m^3}{r} e^{-r/2m} dUdV + r^2 d\sigma^2 \tag{12}$$

where $U = e^{\frac{r+t}{2}}$, $V = e^{\frac{r-t}{2}}(r-1)$ (Kerr, 2023)

One can refer to Kerr's paper for the calculations, Then the author shows that the Jacobian matrix for the map from coordinates (r,t) to (U,V) is analytic and so is its inverse, from (U,V) to (r,t). Therefore, the transformation of coordinates from Eddington to Kruskal is analytic. The same cannot be said when going from Schwarzschild coordinates to Kruskal. Kerr states that the Kruskal extension has no physical significance.

This section in Kerr's paper focuses deeply on the physical relevance of Kruskal rather than a mathematical argument. The second singular region appears to be a white hole, generated by a "nonsingular time-reversed object" (Kerr, 2023) at $r = 0$. Another important point mentioned by the author is that in the real universe, black holes are formed via the immense gravitational collapse of stars and do not originate from white holes like Kruskal. This further signifies that Kruskal has limited physical significance. Although it is mathematically elegant, it does not affect the nature of singularities.

The author also considers the possibility of a nonsingular spherically symmetrical star at the center of Eddington bounded by radius $r = r_0$. The incoming principal null vectors will be radial geodesics that will pass through the central point of the star and terminate on the opposite side. The affine length, given r is an affine parameter for the PNVs, from crossing the horizon would be finite. Again here, we are considering a nonsingular object so the existence of FALLs once again has no relation to infinite curvature. If infinite curvature exists it is due to the yet-unknown physics of the star under those conditions.

Kerr Metric

The Kerr metric is discussed in depth in the original author's paper. Therefore we will cover only the significant results and arguments shown by Kerr. In his analysis, Kerr considers the PNVs on the symmetry axis. Here, we take two sets of light rays: ingoing and outgoing. By calculating the metric we get the result,

$$\frac{dr}{dt} = -1,$$

for the incoming PNV. The author calls this geodesic the "fast" null geodesic. And we also get the result,

$$\frac{dr}{dt} = \frac{r^2 - 2mr + a^2}{r^2 + 2mr + a^2},$$

for the outgoing PNV. This is referred to as the "slow" geodesic. The slow geodesic does not cross either horizon of Kerr. It tends to the outer horizon as $t \longrightarrow -\infty$ and the inter horizon as $t \longrightarrow +\infty$. The radial distance, r, is an affine parameter, and the affine length of the slow null geodesic is finite, with the length being equal to $2\sqrt{m^2 - a^2}$. The same occurs for the slow (outgoing) ray approaching from outside the outer horizon or inside the inner horizon, there is no penetration into the event shell. We are aware that there does not exist a singularity in the Kerr metric other than the ring singularity inside the inner horizon. There exists no singularity in the event shell. Yet there exists a FALL. Hence as discussed in a previous section, the existence of FALLs is not proof of the existence of singularities. This contradicts the Penrose singularity theorem.

As for the fast-moving axial null geodesics, they pass through the ring singularity present inside the inner horizon into the "nonphysical" (Kerr, 2023) branch of Kerr. This region is the supposed "anti-verse" or parallel universe.

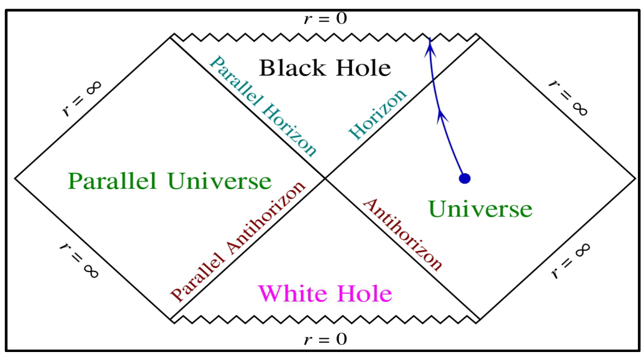

Figure 6. Penrose diagram of the black hole spacetime. (credits: JILA)

The original author believes that this "parallel universe" has no physical significance. When the ring singularity is replaced by a nonsingular source, like a very dense neutron star, the fast(incoming) PNV passes through the body and continues on the other side. On the other side of the body, the affine parameter, r, of the former fast-moving geodesic would be bounded and it will asymptote at the inner horizon. It would become a slow(outgoing) geodesic on the other side. There exists no relation between FALLs and singularities as we replaced the ring with a nonsingular source to obtain this result.

Conclusion

From Kerr's mathematical and logical analysis, we understand the proof presented by Penrose, Hawking, and others is not conclusive of singularities. Singularities must be defined by a region or place where the metric or curvature tensor is either unbounded or not suitably differentiable (Kerr, 2023). It was made clear through the analysis that there does not exist a significant enough relation between FALLs and the above. The singular ring inside of the Kerr metric is just a mathematical convenience and should be replaced by an appropriate star.

The interiors of black holes cannot be explained by mathematics alone and new physics is required to understand their structure. Although, prominent physicists like Roger Penrose and Stephen Hawking must have been aware of this. Their work intends to show the limitations of general relativity. Whether singularities actually do exist will probably be known once General relativity and Quantum mechanics are combined to obtain a theory that explains the behavior of gravity at such high densities. This does not mean that research is impossible before that–as Kerr shows. We must keep questioning and not merely accept things as facts all the time.

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