Mathematical Modeling of the Optimal Acceleration Angle to Avoid Collisions in Space

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ABSTRACT

In outer space, a spaceship experiences almost no external forces that can slow down its movement. Under these circumstances, crashing into other space objects becomes particularly threatening, as the spaceship must move, turn its thrusters, and accelerate in the opposite direction to avoid collision. However, the possible directions of the thrusters that achieve the goal of avoiding an object vary greatly. The goal of this research is to create a mathematical model using knowledge of physics and mathematics to find the optimal angle to accelerate at with a given initial velocity, distance to the object, and maximum acceleration, which will assist spaceships in steering away from objects in the safest way possible. Specifically, this means the angle at which the minimum distance between the spaceship and the object's center is the furthest out of all angles. The mathematical model in the research creates two equations that provide the optimal angle for two separate situations between the given variables and confirm that velocity, distance, and acceleration all play a role in the final optimal angle. The model also provides two equations for the minimum distance that the spaceship will be from the object's center if it accelerates at that angle. This knowledge and model can be used in spaceships to not only avoid collisions against other objects in space but also to leave a restricted zone or a harmful area in space in the safest way possible, potentially saving the lives of passengers from fatal crashes and other threats.

Objective:

The overall objective of this research is to discover how the initial velocity of the spaceship, the distance to the center of the object, and the maximum acceleration possible by the spaceship affect the optimal acceleration angle of the spaceship. The investigation will explore whether all three factors play a role in the final optimal angle and, if so, determine in what ways each factor influence the optimal avert angle. The project aims to design a mathematical model that can calculate this angle in any situation where the three independent variables are positive real numbers. Furthermore, the project seeks to establish additional equations to determine the minimum distance that the spaceship will be from the center of the object, on the trajectory of accelerating away from the object at the optimal avert angle.

Introduction:

Humans have always looked up at the night sky and imagined the endless possibilities in space. The age of space exploration started during the mid-twentieth century, when rapidly developing technologies brought the first artificial satellite and the first moon landing. Since then, humans have launched thousands of satellites into space to orbit Earth and dozens more to the Moon, Mars, Venus, and deep space.

Space exploration improves the lives of all people by advancing new technologies, expanding the economy, and creating new career paths. The development and commercial use of satellites have brought the world together through the internet and allowed every country to reach unprecedented levels of convenience and efficiency. The venture into unknown lands also provides unlimited inspiration and ambition for a new generation of scientists and engineers. In the world of science, space exploration aims to answer questions as practical as weather formation and as abstract as the origin of the universe. Finally, development in space exploration requires the collaborative effort of all of mankind, bringing countries across borders in harmony and collaboration to all work together and venture into the vast unknown.

When exploring outer space, however, spaceships must survive the constant risk of crashing into other objects. Down on the ground, transportation vehicles enjoy the luxury of having both a road and a tire with an immense amount of friction. Cars, bicycles, and other land transportation simply need to hold onto their brakes, and the force of friction can slow them to a halt almost instantly. In outer space, however, with almost no external forces other than low amounts of gravitational fields, slowing down and avoiding crashes become far more difficult. As Newton's first law states, an object in motion will stay in motion until another force acts on it. Therefore, even if the spaceship has no engines activated, it can nevertheless fly at objects at fatal speeds. This is especially true when moving in an orbit around a moon, planet, or star, where spaceships must orbit at a certain speed to counteract the gravitational force with centrifugal force. To avoid a collision, a spaceship must launch its thrusters in a different direction from the object and change its path to deviate further and further away from the object.

Currently, the main sources of collision threats for satellites include space debris and other satellites. As humans launch more and more satellites into space, the materials that are ejected in the launching process or obsolete satellites often remain orbiting around Earth. This poses a significant threat to future satellites and restricts their orbiting paths. For instance, in 2019, a satellite from the European Space Agency had to perform a collision avoidance maneuver to avoid crashing into an obsolete Starlink satellite by changing its orbit with thrusters. As space technology develops and commercializes, however, sources of collision threats broaden greatly, such as flying into planets and moons, into space stations, into other spaceships, and much more.

This research project focuses on situations where the spaceship is flying towards a large object, such as a space station, where it must exert all its possible energy to avoid a crash. Since the outcome of the avoidance maneuver determines the survival of the entire spaceship and all passengers inside, the solution proposed to avoid crashes focuses solely on the effectiveness of the collision avoidance maneuver. Consequently, the model does not consider the efficient use of fuel, like maneuvers used to avoid small space debris. Since the maximum acceleration of the spaceship, initial velocity, and distance to the object cannot be decided by the spaceship during the crash, the mathematical model created in this project proposes the best angle for the spaceship to turn towards to avoid crashing onto the object, which is the one variable spaceships can control with rotation thrusters.

Hypothesis:

The main hypothesis is that the optimal angle of acceleration is not a fixed value that applies to all situations, and that the distance to the object, the initial velocity of the spaceship, and the maximum acceleration of the spaceship will all impact the final optimal angle. This hypothesis can be justified by using two extreme situations: If the spaceship is moving at a slow initial speed relative to the maximum acceleration of the thrusters, the optimal angle would be directly opposite to velocity, where the spaceship will quickly slow down from the much stronger acceleration. On the contrary, if the spaceship was moving at extremely high speeds compared to acceleration, it would crash into the object if it was accelerating in the same direction as in the example before. Alternatively, by positioning the thrusters closer toward perpendicular to the velocity direction, the spaceship can gain some distance before approaching the object by deviating away from its own path towards the object.

Another hypothesis for this mathematical model is that above a certain situation between acceleration, velocity, and distance, and to the extreme of that situation, the optimal avert angle would be fixed to 0°, and another situation would fix the angle to 90°. This can be shown by the extreme examples provided in the main hypothesis as well, where the two situations led to the optimal angle being close to 0° and 90°. Therefore, if the relationship between the three independent variables was compared to a number line, then the hypothesis is that in the model, the optimal avert angle moves between 0˚ and 90˚ on a certain segment of the line, while from the two ends of the segment approaching positive and negative infinity, the angle is fixed at 0˚ and 90˚.

Equations

Below are some mathematical equations used in the model that help calculate and prove the optimal angle throughout the model.

Equation 1: Distance Formula
The distance formula states that on a two-dimensional plane, the distance between two points (x_1, y_1) and (x_2, y_2), using the Pythagorean theorem, can be expressed as:

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Equation 2: Physics equation of motion
The physics formula for the displacement of an object over time (t) with a given initial distance (d), initial velocity (v) of the object, and acceleration (a) of the object can be written as:

$$distance = d + vt + 0.5at^2$$

Equation 3: Vector Resolution Equation
The vector resolution equation states that a vector with a magnitude of $\|v\|$ and an angle of θ can be written as the sum of the two following vectors in component form:

$$\langle \|v\| \cos\theta, 0\rangle + \langle 0, \|v\| \sin\theta\rangle$$

Equation 4: Pythagorean identity
Derived from the Pythagorean theorem, one identity from trigonometric functions frequently used in the model is that for any angle θ, the following equation is true:

$$(\cos\theta)^2 + (\sin\theta)^2 = 1$$

Equation 5-10: Differentiation Rules
The mathematical model also uses a variety of basic derivative rules, including:

- Constant rule: $\frac{dy}{dx} c = 0$
- Sum rule: $\frac{dy}{dx}(f(x) + g(x)) = f'(x) + g'(x)$
- Product rule: $\frac{dy}{dx}(f(x) \times g(x)) = f'(x) \times g(x) + g'(x) \times f(x)$
- Power rule: $\frac{dy}{dx}(x^a) = (ax^{a-1})$
- Chain rule: $f(x) = g\big(h(x)\big), f'(x) = g'(h(x)) \times h'(x)$
- Trigonometric derivatives rule: $\frac{dy}{dx}\cos(x) = -\sin(x)$, $\frac{dy}{dx}\sin(x) = \cos(x)$

The Mathematical Model

Variables Used

- Initial distance = d
- Initial velocity = v
- Maximum acceleration = a

- Angle of acceleration = A (0˚ - 90˚)
- Time = t

The domain for all the variables other than A is all positive real numbers.

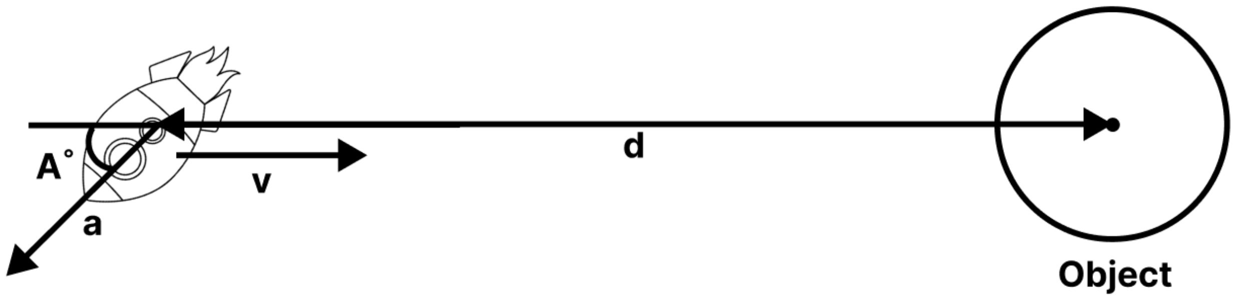

Figure 1 – Mathematical Model Diagram. This diagram depicts the proposed situation in this investigation of a spaceship about to crash onto an object, with all the variables used in the situation labeled.

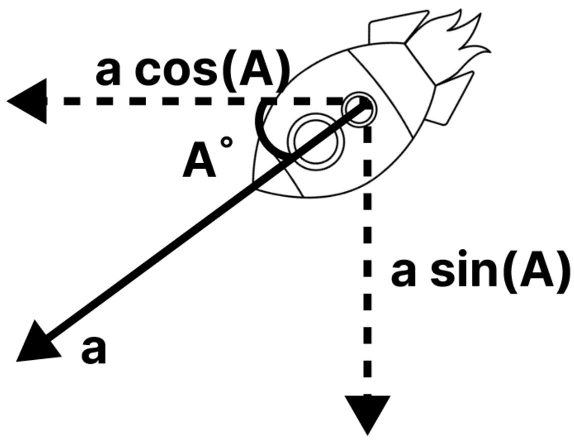

Figure 2 – Spaceship Force Analysis. Vector diagram separating the spaceship's acceleration vector into horizontal and vertical vectors with the vector resolution equation.

Solving for Optimal Avert Angle

Function of horizontal (X) and vertical (Y) distances to the object's center over time (t)

$$X = \sqrt{(0.5t^2 a\cos(A) - tv + d)^2} \qquad Y = 0.5t^2 a\sin(A)$$

Function of distance (y) to the object's center over time (x) using the Pythagorean theorem

$$y = \sqrt{(0.5x^2 a\cos(A) - xv + d)^2 + (0.5x^2 a\sin(A))^2}$$

The optimal avert angle would be where the minimum distance to the collision object is the furthest out of all possible angles from 0˚ to 90˚, to stay as far away from the object as possible. To start in solving this, the minimum of the function must first be solved. To do this, the distance function can be differentiated and find the x-value where the derivative equals 0.

Differentiating the Distance Function

$$y = \sqrt{(0.5x^2 a\cos(A) - xv + d)^2 + (0.5x^2 a\sin(A))^2}$$

$$y' = \frac{\frac{dy}{dx}((0.5x^2 a\cos(A) - xv + d)^2 + (0.5x^2 a\sin(A))^2)}{2\sqrt{(0.5x^2 a\cos(A) - xv + d)^2 + (0.5x^2 a\sin(A))^2}}$$

$$y' = \frac{\frac{dy}{dx}(0.5x^2 a\cos(A) - xv + d)^2 + \frac{dy}{dx}(0.5x^2 a\sin(A))^2}{2\sqrt{(0.5x^2 a\cos(A) - xv + d)^2 + (0.5x^2 a\sin(A))^2}}$$

$$y' = \frac{2(0.5x^2 a\cos(A) - xv + d)\frac{dy}{dx}(0.5x^2 a\cos(A) - xv + d) + \frac{dy}{dx}(0.5x^2 a\sin(A))^2}{2\sqrt{(0.5x^2 a\cos(A) - xv + d)^2 + (0.5x^2 a\sin(A))^2}}$$

$$y' = \frac{2(0.5x^2 a\cos(A) - xv + d)(xa\cos(A) - v) + \frac{dy}{dx}(0.5x^2 a\sin(A))^2}{2\sqrt{(0.5x^2 a\cos(A) - xv + d)^2 + (0.5x^2 a\sin(A))^2}}$$

$$y' = \frac{2(0.5x^2 a\cos(A) - xv + d)(xa\cos(A) - v) + 2(0.5x^2 a\sin(A))\frac{dy}{dx}(0.5x^2 a\sin(A))}{2\sqrt{(0.5x^2 a\cos(A) - xv + d)^2 + (0.5x^2 a\sin(A))^2}}$$

$$y' = \frac{2(0.5x^2 a\cos(A) - xv + d)(xa\cos(A) - v) + 2(0.5x^2 a\sin(A))\,(xa\sin(A))}{2\sqrt{(0.5x^2 a\cos(A) - xv + d)^2 + (0.5x^2 a\sin(A))^2}}$$

$$y' = \frac{2(0.5x^2 a\cos(A) - xv + d)(xa\cos(A) - v) + (x^2 a\sin(A))\,(xa\sin(A))}{2\sqrt{(0.5x^2 a\cos(A) - xv + d)^2 + (0.5x^2 a\sin(A))^2}}$$

$$y' = \frac{2(0.5x^2 a\cos(A) - xv + d)(xa\cos(A) - v) + (x^3\,a^2\,\sin^2(A))}{2\sqrt{(0.5x^2 a\cos(A) - xv + d)^2 + (0.5x^2 a\sin(A))^2}}$$

This is the derivative of the distance function. When the derivative of the function passes through the x axis, or when the y-value is 0, the original function is at its minimum.

Finding the X-axis Intersection of the Function

$$0 = \frac{2(0.5x^2 a\cos(A) - xv + d)(xa\cos(A) - v) + (x^3\,a^2\,\sin^2(A))}{2\sqrt{(0.5x^2 a\cos(A) - xv + d)^2 + (0.5x^2 a\sin(A))^2}}$$

$$0 = 2(0.5x^2 a\cos(A) - xv + d)(xa\cos(A) - v) + x^3\,a^2\,\sin^2(A))$$

$$0 = (x^2 a\cos(A) - 2xv + 2d)(xa\cos(A) - v) + x^3\,a^2\,\sin^2(A)$$

$$0 = xa\cos(A)\,(x^2 a\cos(A) - 2xv + 2d) - v(x^2 a\cos(A) - 2xv + 2d) + x^3\,a^2\,\sin^2(A)$$

$$0 = x^3 a^2\cos^2(A) - 2x^2 av\cos(A) + 2xad\cos(A) - x^2 av\cos(A) + 2xv^2 - 2dv + x^3\,a^2\,\sin^2(A)$$

$$0 = x^3(a^2\cos^2(A) + a^2\sin^2(A)) - x^2(2av\cos(A) + av\cos(A)) + x(2ad\cos(A) + 2v^2) - 2dv$$

$$0 = x^3(a^2\,(\cos^2(A) + \sin^2(A))) - x^2(3av\cos(A)) + x(2ad\cos(A) + 2v^2) - 2dv$$

$$0 = x^3(a^2) - x^2(3av\cos(A)) + x(2ad\cos(A) + 2v^2) - 2dv$$

Above is a cubic equation that finds the minimum point of the distance function.

$$y = \sqrt{(0.5x^2a\ \cos(A) - xv + d)^2 + (0.5x^2a\ \sin(A))^2}$$

We now return to the distance function. Here, when the function is solved for when $x = \frac{d}{v}$, all the trigonometric functions are cancelled out.

Solving for the Distance Function for $x = \frac{d}{v}$

$$y = \sqrt{(0.5x^2a\ \cos(A) - xv + d)^2 + (0.5x^2a\sin(A))^2}$$

$$y = \sqrt{(0.5(\frac{d^2}{v^2})a\cos(A) - v(\frac{d}{v}) + d)^2 + (0.5(\frac{d^2}{v^2})a\sin(A))^2}$$

$$y = \sqrt{(0.5\frac{d^2}{v^2}\ a\cos(A) - d + d)^2 + (0.5\frac{d^2}{v^2}a\sin(A))^2}$$

$$y = \sqrt{(0.5\frac{d^2}{v^2}\ a\cos(A))^2 + (0.5\frac{d^2}{v^2}a\sin(A))^2}$$

$$y = \sqrt{(\frac{ad^2\cos(A)}{2v^2})^2 + (\frac{ad^2\sin(A)}{2v^2})^2}$$

$$y = \sqrt{\frac{a^2d^4\cos^2(A)}{4v^4} + \frac{a^2d^4\sin^2(A)}{4v^4}}$$

$$y = \sqrt{\frac{a^2d^4}{4v^4}(\cos^2(A) + \sin^2(A))}$$

$$y = \sqrt{\frac{a^2d^4}{4v^4}}$$

$$y = \frac{ad^2}{2v^2}$$

When $x = \frac{d}{v}$, the angle terms in the function are cancelled out completely. This means that at this x-value, the angle of acceleration does not affect the distance, and regardless of the angle, the distance at this point will always be $\frac{ad^2}{2v^2}$

Since this point remains constant and the distance function must pass through it, the optimal angle of acceleration would be when this point, $x = \frac{d}{v}$, is the minimum point of the function. Therefore, the cubic derivative equation previously solved can be used to calculate what angle A will $x = \frac{d}{v}$ be equal to 0, or the minimum of the original function.

Solving for the Angle A where (Derivative of the Point $x = \frac{d}{v}$) = 0

$$x^3(a^2) - x^2(3av\cos(A)) + x(2v^2 + 2ad\cos(A)) - 2dv = 0$$

$$\frac{d^3}{v^3}(a^2) - \frac{d^2}{v^2}(3av\cos(A)) + \frac{d}{v}(2v^2 + 2ad\cos(A)) - 2dv = 0$$

$$\frac{a^2d^3}{v^3} - \frac{3ad^2v\cos(A)}{v^2} + \frac{2dv^2 + 2ad^2\cos(A)}{v} - 2dv = 0$$

$$\frac{a^2d^3}{v^3} - \frac{3ad^2\cos(A)}{v} + \frac{2dv^2}{v} + \frac{2ad^2\cos(A)}{v} - 2dv = 0$$

$$\frac{a^2d^3}{v^3} - \frac{3ad^2\cos(A)}{v} + 2dv + \frac{2ad^2\cos(A)}{v} - 2dv = 0$$

$$\frac{a^2d^3}{v^3} - \frac{3ad^2\cos(A)}{v} + \frac{2ad^2\cos(A)}{v} = 0$$

$$\frac{a^2d^3}{v^3} + \frac{-3ad^2\cos(A) + 2ad^2\cos(A)}{v} = 0$$

$$\frac{a^2d^3}{v^3} + \frac{-ad^2\cos(A)}{v} = 0$$

$$\frac{a^2d^3}{v^3} = \frac{ad^2\cos(A)}{v}$$

$$\frac{a^2d^3}{v^2} = ad^2\cos(A)$$

$$\frac{ad}{v^2} = \cos(A)$$

$$A = \cos^{-1}\left(\frac{ad}{v^2}\right)$$

This formula gives the optimal avert angle with a given acceleration, distance, and initial velocity. However, the maximum domain of the $\cos^{-1}$ function is 1, and $\frac{ad}{v^2}$ might be larger than 1. In that case, the minimum point of the distance function cannot be set to $x = \frac{d}{v}$ by changing the angle between 0° - 90°. Therefore, this final equation only solves the optimal avert angle partially, within a certain situation, and the other portion remains to be solved.

Solving for Optimal Avert Angle when $\frac{ad}{v^2} > 1$

Even when $\frac{ad}{v^2}$ is outside of the domain of $\cos^{-1}$, the point mentioned previously where $x = \frac{d}{v}$ still does not change regardless of the angle. To find the optimal avert angle when $\frac{ad}{v^2} > 1$, the relationship between the minimum point of the function in this situation and the fixed point at $x = \frac{d}{v}$ must first be established. Below is the proof that for any angle in the distance function where $\frac{ad}{v^2} > 1$, $\frac{d}{v}$ will always be larger than the x-coordinate of the distance function's minimum. As mentioned, when the derivative of the distance function reaches 0, the distance function is at its minimum point. Since, after the minimum point, the rocket will continue to accelerate further away from the collision object, the derivative will continue to increase to a positive y-value. Therefore, if at the point $x = \frac{d}{v}$, the y-value of the derivative is positive, this means $\frac{d}{v}$ is larger than the x-value of the derivative's x-axis intercept, and thus $\frac{d}{v}$ is larger than the x-value of the minimum point. These calculations are almost identical to the calculations done above, except instead of making the equation equal to zero it is now greater than zero.

Proving ($x = \frac{d}{v}$) >0 on the Distance Function's Derivative when $\frac{ad}{v^2} > 1$

$$x^3(a^2) - x^2(3av\cos(A)) + x(2v^2 + 2ad\cos(A)) - 2dv > 0$$

$$\frac{d^3}{v^3}(a^2) - \frac{d^2}{v^2}(3av\cos(A)) + \frac{d}{v}(2v^2 + 2ad\cos(A)) - 2dv > 0$$

$$\frac{a^2d^3}{v^3} - \frac{3ad^2v\cos(A)}{v^2} + \frac{2dv^2 + 2ad^2\cos(A)}{v} - 2dv > 0$$

$$\frac{a^2d^3}{v^3} - \frac{3ad^2\cos(A)}{v} + \frac{2dv^2 + 2ad^2\cos(A)}{v} - 2dv > 0$$

$$\frac{a^2d^3}{v^3} - \frac{3ad^2\cos(A)}{v} + 2dv + \frac{2ad^2\cos(A)}{v} - 2dv > 0$$

$$\frac{a^2d^3}{v^3} - \frac{3ad^2\cos(A)}{v} + \frac{2ad^2\cos(A)}{v} > 0$$

$$\frac{a^2d^3}{v^3} + \frac{-3ad^2\cos(A) + 2ad^2\cos(A)}{v} > 0$$

$$\frac{a^2d^3}{v^3} + \frac{-ad^2\cos(A)}{v} > 0$$

$$\frac{a^2d^3}{v^3} > \frac{ad^2\cos(A)}{v}$$

$$\frac{a^2d^3}{v^2} > ad^2\cos(A)$$

$$\frac{ad}{v^2} > \cos(A)$$

As shown here, when $x = \frac{d}{v}$ is substituted into the derivative of the distance function with the inequality that the derivative at this point is greater than 0, after simplification, the result is that $\cos(A)$ is less than $\frac{ad}{v^2}$. The condition before this calculation is that $\frac{ad}{v^2} > 1$, and since the maximum of $\cos(A)$ is 1, this final inequality is true regardless of the angle. Therefore, when $\frac{ad}{v^2} > 1$, the derivative will always intersect the line $x = \frac{d}{v}$ at a positive y-value, and the x-value of the fixed point at $x = \frac{d}{v}$ will always be larger than the x-value of the distance function's minimum.

The relationship between the $x = \frac{d}{v}$ point and the minimum has now been established. Returning once again to the distance function, where the terms can be expanded so that only the cosine trigonometric function remains using the Pythagorean identity:

Expanding the Distance Function

$$y = \sqrt{(0.5x^2\,\mathrm{a}\cos(A) - xv + d)^2 + (0.5x^2 a\sin(A))^2}$$

$$y = \sqrt{0.25x^4\cos^2(A)a^2 + x^2v^2 + d^2 + 2(0.5x^2ad\cos(A) - 0.5x^3av\cos(A) - xdv) + 0.25x^4\sin^2(A)a^2}$$

$$y = \sqrt{0.25x^4a^2(\cos^2(A) + \sin^2(A)) + x^2v^2 + d^2 + 2(0.5x^2ad\cos(A) - 0.5x^3av\cos(A) - xdv)}$$

$$y = \sqrt{0.25x^4a^2 + x^2v^2 + d^2 + 2(0.5x^2ad\cos(A) - 0.5x^3av\cos(A) - xdv)}$$

$$y = \sqrt{0.25x^4a^2 + x^2v^2 + d^2 + x^2ad\cos(A) - x^3av\cos(A) - 2xdv}$$

$$y = \sqrt{0.25x^4a^2 - x^3av\cos(A) + x^2(v^2 + ad\cos(A)) - 2xdv + d^2}$$

Below is the proof that when $\frac{ad}{v^2} > 1$, as the angle A increases from 0° to 90°, all distances between x-values of $0 \leq x < \frac{d}{v}$ constantly decreases. As proven above, this domain of x includes the minimum point of the distance function.

To investigate the change in the angle A that applies to all values of x within that boundary, x can now be substituted with a variable "b" and the range of "b" can be set as $0 \leq b < \frac{d}{v}$. Angle A can then be used as the input of the function by replacing A in the cosine function with x, and setting the domain of the function to be $0° < x < 90°$, written as the function below:

Modified Function with Angle as the Input

$$y = \sqrt{0.25b^4a^2 - b^3av\cos(x) + b^2(ad\cos(x) + v^2) - 2bdv + d^2}$$

To prove that as the angle A, or in this new function, x, increases, the y-value decreases, the new function can first be differentiated, finding the rate of change. If the derivative is always negative between the domain $0° < x < 90°$, this means that the y-value is constantly decreasing as the angle A is increasing.

Proving of Modified Function's Derivative as Negative for $0^\circ < x < 90^\circ$

$$y = \sqrt{0.25b^4a^2 - b^3av\cos(x) + b^2(ad\cos(x) + v^2) - 2bdv + d^2}$$

$$y = \sqrt{0.25b^4a^2 - b^3av\cos(x) + b^2ad\cos(x) + b^2v^2 - 2bdv + d^2}$$

$$y' = \frac{\frac{dy}{dx}(0.25b^4a^2 - b^3av\cos(x) + b^2ad\cos(x) + b^2v^2 - 2bdv + d^2)}{2\sqrt{0.25b^4a^2 - b^3av\cos(x) + b^2ad\cos(x) + b^2v^2 - 2bdv + d^2}}$$

$$y' = \frac{\frac{dy}{dx}0.25b^4a^2 - \frac{dy}{dx}b^3av\cos(x) + \frac{dy}{dx}b^2ad\cos(x) + \frac{dy}{dx}b^2v^2 - \frac{dy}{dx}2bdv + \frac{dy}{dx}d^2}{2\sqrt{0.25b^4a^2 - b^3av\cos(x) + b^2ad\cos(x) + b^2v^2 - 2bdv + d^2}}$$

$$y' = \frac{b^3av\sin(x) - b^2ad\sin(x)}{2\sqrt{0.25b^4a^2 - b^3av\cos(x) + b^2ad\cos(x) + b^2v^2 - 2bdv + d^2}}$$

Now that the derivative of the function has been found, it can now be proven that the derivative is always negative from 0˚ to 90˚.

$$\frac{b^3av\sin(x) - b^2ad\sin(x)}{2\sqrt{0.25b^4a^2 - b^3av\cos(x) + b^2ad\cos(x) + b^2v^2 - 2bdv + d^2}} < 0$$

$$b^3av\sin(x) - b^2ad\sin(x) < 0$$

$$b^3av\sin(x) < b^2ad\sin(x)$$

$$b^3av < b^2ad$$

$$bv < d$$

$$b < \frac{d}{v}$$

Since the boundary of "b" when modifying the function is that $0 \leq b < \frac{d}{v}$, this final inequality is always true, and therefore the derivative of the modified function is always negative. This means the y-value is always decreasing as the angle increases, and for all situations where $\frac{ad}{v^2} > 1$ the minimum point decreases as the angle increases.

Since the optimal avert angle maximizes the minimum point's distance, and as the angle increases the distance to the object at the minimum point decreases, this means that the optimal avert angle in all situations where $\frac{ad}{v^2} > 1$ is A = 0˚, where the angle is at its minimum and the minimum point's distance is at its maximum.

Calculating Minimum Distance

Now that the optimal avert angles have been found, the minimum distance between the spaceship and the collision object's center can also be calculated at that angle.

As mentioned, for all situations where $\frac{ad}{v^2} \leq 1$, where the function $\cos^{-1}(\frac{ad}{v^2})$ can be applied for the optimal avert angle, the minimum distance will be at $x = \frac{d}{v}$, when y = $\frac{ad^2}{2v^2}$

For the situations where $\frac{ad}{v^2} > 1$, the minimum angle A=0˚ can be substituted back into the distance formula, where the y-distance term is eliminated, to calculate the minimum distance:

$$y = \sqrt{(0.5x^2a\cos(0) - vx + d)^2 + (0.5x^2a\sin(0))^2}$$

$$y = \sqrt{(0.5x^2a - vx + d)^2}$$

$$y = 0.5x^2a - vx + d$$

The distance formula, when A= 0˚, is now a quadratic equation, in which case the minimum x-value is $-\frac{b}{2a}$

$$minimum\ x - value = -\frac{b}{2a} = -\frac{-v}{2 \times 0.5a} = \frac{v}{a}$$

$$\min = 0.5a\frac{v^2}{a^2} - \frac{v^2}{a} + d$$

$$\min = \frac{v^2}{2a} - \frac{2v^2}{2a} + d$$

$$\min = d - \frac{v^2}{2a}$$

Results

When a spaceship is flying towards an object, with a given acceleration (a), initial distance (d), and initial velocity (v), the optimal avert angle (A) to avoid that object and the minimum distance to the object at that angle are:

If $\frac{ad}{v^2} \leq 1$:
A = $\cos^{-1}(\frac{ad}{v^2})$, minimum distance = $\frac{ad^2}{2v^2}$

If $\frac{ad}{v^2} > 1$:
A = 0˚, minimum distance = $d - \frac{v^2}{2a}$

Graphs of Results

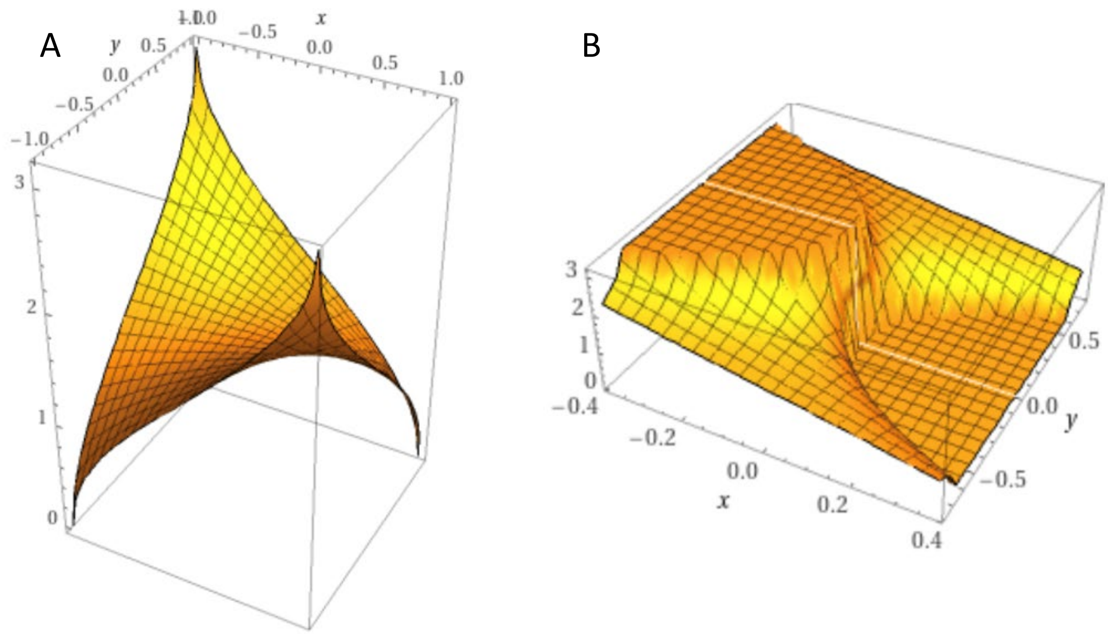

Figure 3 – 3D Visualizations of the Optimal Avert Angle. Figure 3A shows the optimal avert angle on the Z axis in radians, with acceleration (a) and distance (d) as the X and Y axes and a fixed velocity. Figure 3B shows the optimal avert angle on the Z axis in radians, with either acceleration (a) or distance (d) as the X axis and the other as fixed, while the Y axis is the initial velocity (v). Note that only the positive x- and y-values (right side quadrant) apply in both diagrams.

Conclusion

Real-World Applications

The optimal avert angle can have a variety of potentially life-saving applications in the future of space travel, where traveling to outer space is more and more commercialized and frequent. In space, there are a lot of large objects that spaceships must try to avoid, including space stations, space cities, other spaceships, and natural bodies such as asterioids, moons, planets, and stars. When a spaceship realizes a potential crashing threat, it can use this model to find the best way to avoid the object, distancing itself from the crashing threat as far as possible. At high speeds, this angle could determine the survival or destruction of the spaceship.

As space travel develops, there will also be more situations where the angle is critically important. For instance, in the future, there might be certain restricted zones similar to restricted areas on earth. If a driver suddenly realizes the spaceship is flying into one of such zones, the driver must turn the spaceship as fast as possible to avoid the consequences of flying into the zone, in which case the angle is also helpful. Additionally, if a spaceship is flying into an object that is about to explode or an object releasing harmful radiation, the damage taken by the spaceship is inversely proportional to the distance to that object, in which case the angle can minimize the damage.

Furthermore, although the model portrays the crash object as a point in space, in reality, the object would have a certain radius from its center. Consequently, calculating the minimum distance to the object would also be critically important. If the minimum distance is less than the radius of the object, then the spaceship would crash into the object even when using the best angle possible. Therefore, with the equations presented in the model, the spaceship can predict whether it will survive or be destroyed in the crash avoidance maneuver. If it cannot successfully avoid the object, then it must take further emergency precautions, such as ejecting the passengers or sending out a rescue signal. In short, both the equations for the optimal avert angle and the minimum distance are highly valuable for the survival of spaceships and passengers during emergency situations in space.

Reflection on the Objective

The objective of this investigation was to develop a model to determine the optimal angle to accelerate at to avoid an object with a given acceleration, distance, and velocity, where all three variables are positive real numbers. Furthermore, the project aimed to calculate the minimum distance between the spaceship and the object's center. Reflecting on the objective, both goals were clearly met in this investigation. The inverse cosine function in the result can calculate an angle for any angle with a, d, and v as positive real numbers within 1, and the optimal angle for any value of $\frac{ad}{v^2}$ greater than 1 has also been found to be 0˚. With the given equations, the investigation also found the two equations for the two separate situations to determine the minimum distance between the spaceship and the object's center. As mentioned in the introduction, the model also focused solely on maximizing distance, disregarding fuel consumption and energy efficiency, as it is mainly used when the safety of the spaceship is at risk.

Reflection on the Hypothesis

The first hypothesis for this experiment was that velocity, acceleration, and distance would all play a factor in the final optimal angle of the spaceship, justified using two extreme situations between the three variables. The final equation does support this hypothesis, as all three variables are used in the equation to determine the final angle with the inverse cosine function. The equation also supports the two justifications previously mentioned to support the hypothesis. For instance, the hypothesis stated that if the velocity was much higher relative to acceleration and distance, the optimal angle would be more vertical, deviating from the path instead of trying to slow the spaceship down. In this situation, the faster the initial velocity, the closer $\frac{ad}{v^2}$ would be to 0. Between 0˚ and 90˚, $\cos^{-1}(0) = 90^\circ$, supporting this statement. On the contrary, when the acceleration is a lot faster relative to velocity, $\frac{ad}{v^2}$ would be closer to or above 1, in which case $\cos^{-1}(1) = 0^\circ$, and any situation above 1 would also have 0˚ as its optimal angle, flying directly toward the object to maximum its distance from the object's center.

The second hypothesis is that if the relationship between a, d, and v is imagined as a number line, then the optimal angle would move from 0˚ to 90˚ within a segment of that number line, and from the two end points to the two extremes, the angle would be fixed at 0˚ and 90˚. However, this hypothesis is only partially supported. When $\frac{ad}{v^2}$ is any value greater than 1, the optimal angle is fixed at 0˚. Therefore, this portion supports the second hypothesis. However, for the optimal angle to be fixed at 90˚, $\frac{ad}{v^2}$ must be equal to 0, as $\cos^{-1}(0) = 90^\circ$. In the context of this physics situation, neither acceleration nor distance can be equal to 0 for the situation to make sense. The only way for $\frac{ad}{v^2}$ to be equal to 0 is when velocity is infinitely greater than acceleration and distance. Consequently, the optimal angle is only fixed at 90˚ when the initial velocity approaches infinity, and not after a certain point, as suggested by the number line example. Therefore, the second portion of the second hypothesis is not supported by the results.

Limitations and Future Extensions

One major limitation of this model is that it ignores another force that plays a significant role in the trajectory of the spaceship in avoiding collision with large objects: gravity. When escaping an object, the force of gravity would constantly accelerate the spaceship towards its center, fighting against the acceleration of the spaceship. Therefore, a future improvement to the investigation could be to account for the gravity of the object in the model. With gravity included, however, the optimal avert angle would likely be a function over time rather than a constant, as the direction pull of the gravity will constantly change relative to the spaceship. Therefore, the spaceship's thrusters must constantly change their angle to counteract that pull. Nevertheless, this limitation only causes an imperfect estimation when encountering large celestial bodies such as moons, planets, and stars. For most other situations, such as avoiding a certain zone or space station, the gravity of the object is completely insignificant, and the model is still very useful in avoiding those objects.

Another limitation of the model is that it does not account for the time it takes for the spaceship to rotate itself with its thrusters to the optimal angle. This is another factor that could play a role in the model for the most precise angle possible. When the spaceship rotates, especially if its rotational thrusters are weak, waiting until it reaches the optimal angle would result in a large amount of precious time being lost, which could have been used to start escaping the crash object. Therefore, in a real situation, the best avoidance maneuver would likely be to immediately start accelerating when the spaceship angle is below 135°, when the vertical vector of the acceleration away from the object is greater than the horizontal vector flying towards the object. However, calculating an optimal angle for a situation like that would be far more complex, as it involves the instantaneous rates of change for several factors, which then influence the rates of change for various forces acting on the spaceship. Nevertheless, it remains a potential improvement to this mathematical model.

References

Aerospace Corporation. (2024, March 1). *A brief history of space exploration: The aerospace corporation.* https://aerospace.org/article/brief-history-space-exploration

European Space Agency. (2019, March 9). *ESA spacecraft dodges large constellation*. ESA. http://www.esa.int/Space_Safety/ESA_spacecraft_dodges_large_constellation

NASA. (2023, December 14). *Benefits to humanity*. http://www.nasa.gov/humans-in-space/benefits-to-humanity/

NASA. (2023a, September 22). *Why go to space*. http://www.nasa.gov/humans-in-space/why-go-tospace/

National Geographic. (2023). *The history of space exploration.* https://education.nationalgeographic.org/resource/history-space-exploration/

Slater. (2023). *Collision Avoidance for Satellites in Formation Flight*. Journal of Guidance, Control, and Dynamics. https://arc.aiaa.org/doi/abs/10.2514/1.G003483?journalCode=jgcd